A. A. Himwich, Physician, New York, N. Y.
F. C. Hodgson, Publisher, New York City, N. Y.
F. J. Holder, University of Pittsburgh, Pittsburgh, Pa.
L. S. Hulburt, Johns Hopkins University, Baltimore, Md.
E. V. Huntington, Harvard University, Cambridge, Mass.
W. A. Hurwitz, Cornell University, Ithaca, N. Y.
Dunham Jackson, Harvard University, Cambridge, Mass.
G. H. Jamison, State Normal School, Kirksville, Mo.
S. A. Joffe, Assistant Actuary, New York, N. Y.
Edward Kasner, Columbia University, New York, N. Y.
Edward Kircher, Harvard University, Cambridge, Mass.
E. H. Koch, Jr., High School of Commerce, New York, N. Y.
K. W. Lamson, Graduate School, University of Chicago, Chicago, Ill.
Florence P. Lewis, Goucher College, Baltimore, Md.
G. H. Ling, University of Saskatchewan, Saskatoon, Can.
Joseph Lipka, Massachusetts Institute of Technology, Cambridge, Mass.
L. L. Locke, Brooklyn Training School for Teachers, Brooklyn, N. Y.
W. R. Longley, Yale University, New Haven, Conn.
Emilie N. Martin, Mt. Holyoke College, South Hadley, Mass.
James McClay, Columbia University, New York, N. Y.
Helen A. Merrill, Wellesley College, Wellesley, Mass.
Mansfield Merriman, Consulting Engineer, New York, N. Y.
E. J. Miles, Yale University, New Haven, Conn.
Bessie I. Miller, Rockford College, Rockford, Ill.
G. A. Miller, University of Illinois, Urbana, Ill.
J. A. Miller, Swarthmore College, Swarthmore, Pa.
J. S. Miller, Emory and Henry College, Emory, Va.
H. B. Mitchell, Columbia University, New York, N. Y.
F. M. Morgan, Dartmouth College, Hanover, N. H.
Frank Morley, Johns Hopkins University, Baltimore, Md.
Richard Morris, Rutgers College, New Brunswick, N. J.
B. L. Newkirk, University of Minnesota, Minneapolis, Minn.
E. J. Oglesby, College of William and Mary, Williamsburg, Va.
G. D. Olds, Amherst College, Amherst, Mass.
F. W. Owens, Cornell University, Ithaca, N. Y.
George Paaswell, Civil Engineer, New York, N. Y.
Alexander Pell, South Hadley, Mass.
Anna J. Pell, Mount Holyoke College, South Hadley, Mass.
A. D. Pitcher, Western Reserve University, Cleveland, O.
W. R. Ransom, Tufts College, Tufts College, Mass.
H. W. Reddick, Cooper Union, New York, N. Y.
Emma M. Requa, Hunter College, New York, N. Y.
R. G. D. Richardson, Brown University, Providence, R. I.
H. L. Rietz, University of Illinois, Urbana, Ill.

R. B. Robbins, Yale University, New Haven, Conn.
E. D. Roe, Jr., Syracuse University, Syracuse, N. Y.
R. E. Root, U. S. Naval Academy, Annapolis, Md.
D. A. Rothrock, Indiana University, Bloomington, Ind.
Mary E. Sinclair, Oberlin College, Oberlin, O.
H. E. Slaught, University of Chicago, Chicago, Ill.
Clara E. Smith, Wellesley College, Wellesley, Mass.
D. E. Smith, Columbia University, New York, N. Y.
P. F. Smith, Yale University, New Haven, Conn.
Sarah E. Smith, Mount Holyoke College, South Hadley, Mass.
W. M. Smith, Lafayette College, Easton, Pa.
Jessie Spearing, Graduate School, Columbia University, New York, N. Y.
J. M. Stetson, Western Reserve University, Cleveland, Ohio.
H. D. Thompson, Princeton University, Princeton, N. J.
F. C. Touton, Central High School and Junior College, St. Joseph, Mo.
A. B. Turner, College of the City of New York, New York, N. Y.
J. N. Van der Vries, University of Kansas, Lawrence, Kan.
Oswald Veblen, Princeton University, Princeton, N. J.
Evelyn Walker, Hunter College, New York, N. Y.
C. B. Walsh, Ethical Culture High School, New York, N. Y.
J. H. Weaver, High School, West Chester, Pa.
H. E. Webb, Central High School, Newark, N. J.
Louisa M. Webster, Hunter College, New York, N. Y.
A. L. Wechsler, New York City, N. Y.
Mary E. Wells, Vassar College, Poughkeepsie, N. Y.
E. E. Whitford, College of the City of New York, New York, N. Y.
F. B. Williams, Clark University, Worcester, Mass.
A. H. Wilson, Haverford College, Haverford, Pa.
E. B. Willson, Massachusetts Institute of Technology, Cambridge, Mass.
F. N. Wilson, Princeton University, Princeton, N. J.
J. W. Young, Dartmouth College, Hanover, N. H.
Mabel M. Young, Wellesley College, Wellesley, Mass.

The meeting opened with a joint session of the Association with the American Mathematical Society, Section A of the American Association for the Advancement of Science, and the American Astronomical Society, about two hundred being present. The retiring president of the American Mathematical Society, Professor Ernest W. Brown, of Yale University, gave his presidential address on "The relations of mathematics to the natural sciences." This was followed by the retiring address of Professor Armin O. Leuschner of the University of California, vice-president of Section A of the American Association. In the absence of Professor Leuschner, the address, entitled "Derivation of orbits—theory and practice," was read by Professor M. W. Haskell. This session, like those of Friday and Saturday, was held in Room 301, Hamilton Hall.

A joint dinner of these four organizations was held Thursday evening at the

Park Avenue Hotel. Following the dinner, which was attended by 142 persons, Professor L. P. Eisenhart, vice-president of Section A of the American Association, introduced the following speakers: Professor Florian Cajori, Colorado College; President R. J. Aley, University of Maine; Mr. William Bowie, U. S. Coast and Geodetic Survey; Professor J. A. Miller, Swarthmore College; Mr. G. A. Plimpton, New York City; and Professor Dunham Jackson, Harvard University.

A valuable feature in connection with the meeting was the opportunity afforded on Friday from twelve until two o'clock to visit the famous collection of portraits and medals of mathematicians gathered by Professor David Eugene Smith. The Association is under obligation to Professor Smith.

An appropriate resolution was adopted by the Council, recognizing the courtesy of the department of mathematics of Columbia University, the able work of the program committee in preparing so attractive a program, and the convenience of the facilities provided by the committee on arrangements.

The meetings of the Association continued on Friday morning. The following program was carried out, in accordance with the plan as arranged by the committee under the chairmanship of Professor David Eugene Smith, of Columbia University. At the request of President Hedrick, the chair was occupied for part of Friday morning by Vice-President Huntington and on Saturday morning by President-elect Cajori. Professor J. N. Van der Vries, chairman of the department of institutional delegates, presided at their meeting on Friday afternoon.

Friday Morning.

(1) "Discussions of Fluxions from Berkeley to Woodhouse." PROFESSOR FLORIAN CAJORI, Colorado College.

(2) "University Courses in Mathematics Intended for Teachers of Secondary Mathematics." PROFESSOR M. W. HASKELL, University of California.

(3) Discussion, led by PROFESSOR J. W. YOUNG, Dartmouth College, and PROFESSOR EDWARD KASNER, Columbia University.

Friday Afternoon.

(4) Meeting of Institutional Delegates. (See page 56.)

"A Nucleus for a Mathematical Library." DR. T. H. GRONWALL, New York, N. Y. (Introduced by Professor Oswald Veblen.)

Report of the Library Committee. PROFESSOR W. B. FORD, University of Michigan, Chairman.

Saturday Morning.

(5) "The Mathematics of Aërodynamics." PROFESSOR E. B. WILSON, Massachusetts Institute of Technology.

Discussion by PROFESSOR A. G. WEBSTER, Clark University.

The address of Professor Cajori clothed the dry bones of a somewhat remote controversy in the history of mathematics with an enlivening interest that showed his mastery as a writer of history.

Professor Haskell's paper forms an important contribution to a topic that is at present the object of much study on the part of university and college teachers. It aroused a lively discussion not merely by the two appointed to lead, but by a number of others who either criticized the views of the previous speakers or reported on plans now in operation similar to the one given by Professor Haskell.

In accordance with the policy already established of giving at our meetings one or more reports on the activities of committees of the Association, the Library Committee, appointed in June by President Hedrick, made a report on the plans which it has already formed. The report is printed in full on page 56.

Professor Wilson's paper on Saturday morning added a new phase to the programs of the Association, giving an exposition of the mathematics used in an important and rapidly growing science.

Abstracts are printed below for the papers and the principal discussions, so far as these are available. These are numbered to correspond with the numbers on the program above.

Abstracts of Papers.

(1) Professor Cajori's address, "Discussions of Fluxions from Berkeley to Woodhouse," contained brief accounts of Berkeley's *Analyst* and of three controversies carried on in Great Britain during the eighteenth century on the subject of fluxions. Particular attention was given to Berkeley's rejection of infinitely small quantities, Berkeley's criticism of Newton's derivation of the "moment" of a rectangle, and to Berkeley's "lemma": If in a demonstration an assumption is made, by virtue of which certain conclusions follow, and if afterward that assumption is destroyed or rejected, then all the conclusions that had been reached by the first assumption must also be destroyed or rejected. Berkeley's opponents would not accept this lemma. Robert Woodhouse in 1803 openly and fully accepted it. Rejoinders to Berkeley's *Analyst* were made by James Jurin of Cambridge and John Walton of Dublin. A second controversy on the nature of Newton's fluxions and limit-concept was carried on between James Jurin on one side, and Benjamin Robins and Pemberton on the other. Jurin believed that Newton's variables reached their limits; Robins insisted that a variable cannot reach its limit, and developed the calculus without using infinitely small quantities. This debate constitutes the most thorough discussion of limits of that time. The controversy lasted two years, gave rise to twenty articles which filled over 700 printed pages. Maclaurin's *Fluxions*, 1742, met deservedly with high appreciation, but except in the last part of the text, it did not use any notation for the new analysis; its rhetorical form of exposition made it unattractive reading. The advances in the logical exposition of the calculus in Great Britain were made during the eight years immediately following the publication of the *Analyst*. A third controversy took place in 1850–1852; friends of Emerson were arrayed against friends of Simpson. The discussion was carried on in the *Ladies Diary*, in some other journals less widely known, and in a pamphlet, *Truth Triumphant or Fluxions for the Ladies, showing the cause to be before the effect*, etc. This is the least important of the three controversies.

Towards the latter part of the eighteenth century the efforts at rigorous exposition slackened. Maclaurin was seldom read and Robins was forgotten. The first three editions of the *Encyclopedia Britannica* defined a "fluxion" as an "increment" acquired in "less than any assigned time." Both before and after the period of eight years, 1834–1842, there existed during the eighteenth century in Great Britain a mixture of Continental and British conceptions of the new calculus, a superposition of British symbols and phraseology upon the older Continental concepts. Newton's notation was poor and Leibniz's philosophy of the Calculus was poor. The mixture represented the temporary survival of the least fit of both systems. The subsequent course of events was in the opposite direction; the Leibnizian notation and phraseology was superposed upon the limit-concept as developed by Newton, Jurin, Robins, Maclaurin, D'Alembert and later writers. Woodhouse was the first Englishman to acknowledge openly the great services rendered by Bishop Berkeley in criticizing the philosophical conception of the calculus prevalent at that time. Woodhouse was the forerunner in Cambridge of Babbage, Peacock and the younger Herschel, in the promotion of the principles of pure D-ism in opposition to the dot-age of the university.

(2) Professor Haskell's paper is promised for printing in full in the March issue.

(3) In lieu of an abstract which was not obtainable in time for printing here, Professor Kasner set forth the appreciation of the research aspect of mathematics as the primary requirement of the teacher. This is best acquired by a course in original problems, composed mainly by the students themselves and as closely related to general science and life as possible. New concepts should be introduced by *difficult* interesting problems, easy ones to be discussed later. General scientific method and historical perspective should be emphasized. The teacher's mind must be liberal and flexible; the rough procedure of common sense, the finest logical criticism, and all intermediate stages, should be appreciated sympathetically.

Taking as his point of departure, Professor Haskell's question: "What is geometry?", Professor Young affirmed that teachers of mathematics must make clear to themselves the answer to the question: "What is mathematics?" What is needed is not a formal definition; but a working knowledge, in order that teachers may see clearly what they are trying to accomplish. Judging by prevailing practice in the teaching of algebra, where from 75 per cent. to 90 per cent. of the time is devoted to the development of proficiency in manipulating symbols, one would say that such technical skill is the teacher's conception of what constitutes algebra. Yet no wide-awake teacher would deny that such manipulatory expertness is only an incident and not the real substance of algebra. What then is the real substance of algebra? Courses in algebra must be so reorganized that there would be no need of asking the question. Pending that time, courses for prospective teachers, as outlined by Professor Haskell, must equip the teacher to formulate his answer. It is for this reason that

emphasis on the history of mathematics is justified,—but the history must be taught not primarily to gain information, but rather to develop an insight into the spirit of mathematical enquiry. One result of such insight will be to banish the obsession that mathematical thinking is *deductive*. It is not—every one who has stopped to think about the way he reaches a solution of a mathematical problem knows that in the majority of cases his thinking was not primarily deductive. This obsession is largely responsible for the unnecessary formalism with which our subject is branded. Finally, Professor Young expressed the opinion that sufficient place is not given to the applications of mathematics in Professor Haskell's scheme of courses.

(4) The meeting of institutional delegates is reported below.

(5) Professor Wilson pointed out that there are at least three stages in the growth of a branch of engineering: (1) the initiation by bold adventurers, (2) the development of a mathematical theory, (3) the codification in texts and handbooks. Aëronautical engineering is at present in stage (2), and properly qualified mathematicians like G. H. Bryan and Sir Geo. Greenhill have aided materially in advancing the art. Particular mention was made of the modifications and adaptations of the purely theoretical work which have been introduced by L. Bairstow in England and by J. C. Hunsaker in this country, by virtue of which experimental results may be combined with the methods of mathematicians to develop really accurate principles of airplane design.

Professor Wilson showed that apart from the purely elementary methods of algebra and trigonometry, the branches of mathematics used in aëronautical theory are: (1) The principles of mechanics including moving axes, (2) linear differential equations with constant coefficients, (3) theory of functions of a complex variable. Emphasis was laid on the need of so teaching the theory of functions that it becomes a practically usable branch of mathematics, and of devoting more time to the study of mechanics and mathematical physics, to the end that mathematicians might contribute more to the national defense both in periods of preparation and in crises of adversity.

This paper was discussed by Professor A. G. Webster, of Clark University, in his inimitable manner. The discussion was also participated in by Director L. A. Bauer, of the Carnegie Institution of Washington, and by Dr. W. S. Franklin, of South Bethlehem, Pa. Professor Ernest W. Brown, whose name appeared upon the stated program, was unavoidably prevented from attendance at this session.

Meeting of Institutional Delegates.

The meeting of delegates representing those colleges and universities holding institutional membership was held Friday afternoon at two o'clock. Professor J. N. Van der Vries, of the University of Kansas, had been chosen at the summer meeting as chairman of this department, and as such was in charge of the meeting. It was announced at the morning program that the meeting was to be an open

session and that all were welcome who were interested in the institutional questions which form the special sphere of the activities of this department. The general topic proposed by a special committee for consideration at this meeting was that of mathematical libraries, the best selection of books for a small library of 200, 300, or 500 volumes, and related questions. This topic grew naturally out of the discussion at the summer meeting, and the interest which it evoked is one of several indications of the growing importance attaching to the department of institutional delegates. According to the roll-call made by the secretary, the following 29 institutions were represented by official delegates:

Amherst College, Professor T. C. Esty;
Brooklyn Polytechnic Institute, Dr. J. B. Chittenden;
Brown University, Professor R. G. D. Richardson;
University of Buffalo, Professor W. H. Sherk;
Case School of Applied Science, Professor T. M. Focke;
University of Chicago, Professor H. E. Slaught;
Colorado College, Professor Florian Cajori;
Columbia University, Professor T. S. Fiske;
Cooper Union, Professor H. W. Reddick;
Creighton University, Professor W. F. Rigge;
Dartmouth College, Professor J. W. Young;
Hamilton College, Professor H. S. Brown;
University of Kansas, Professor J. N. Van der Vries;
Kenyon College, Professor R. B. Allen;
Lafayette College, Professor W. M. Smith;
University of Michigan, Professor W. B. Ford;
University of Missouri, Professor E. R. Hedrick;
Mount Holyoke College, Professor Sarah E. Smith;
College of the City of New York, Professor Joseph Allen;
N. Y. State College for Teachers, Professor Harry Birchenough;
New York University, Professor T. W. Edmondson;
Oberlin College, Professor W. D. Cairns;
Princeton University, Professor H. B. Fine;
Rockford College, Professor Bessie I. Miller;
Rutgers College, Professor Richard Morris;
Trinity College, Professor J. D. Flynn;
Wellesley College, Professor Helen A. Merrill;
Wesleyan University, Professor L. A. Howland;
Western Reserve University, Professor A. D. Pitcher.

There were also present members of the faculties of other institutions including the following: Bryn Mawr College, Clark University, Cornell University, Emory and Henry College, Goucher College, Harvard University, Hunter College, University of Indiana, Johns Hopkins University, University of Pittsburgh, Syracuse University, Tufts College, Vassar College, and Yale University.

Inasmuch as the value of Dr. Gronwall's paper consisted most of all in the list of books which he has drawn up, a list which must be seen to be fully appreciated, he did not attempt to read it but his presentation took the form of brief comments upon the principles underlying the selection of the list. Dr. Gronwall's list of books will be printed later, probably in connection with a further report of the Library Committee and at a point where its usefulness will be enhanced by its logical association with two or three other lists now in the process of formation.

Preliminary Report of the Library Committee.

The Library Committee, which consists of Professor W. B. Ford, Chairman, and Professors Florian Cajori, E. S. Crawley, Solomon Lefschetz, W. R. Longley, and R. E. Root, made the following report:

The relation which the Association has, or may be made to have in the near future, to the mathematical libraries of our schools and colleges at once presents a decidedly real and many sided problem. That the Association can render valuable service in this direction was early recognized by the President and others identified with him in the inception of the organization, and later the present committee was asked to arrive if possible at some definite plans for effectively carrying out such work. The committee can as yet make no final report, but it can set forth a number of suggestions which it has received from various sources and of which it fully approves. It is planned to develop these suggestions in the near future so as to put them upon an actual working basis. Meanwhile further suggestions will be most gratefully received, especially from those members who are engaged in schools and colleges where the need of better library advantages is distinctly felt.

The suggestions received are as follows:

1. That the committee prepare certain lists of books and publish them in the Monthly. To be more specific, it has been suggested that lists be prepared suitable for freshmen, sophomores, juniors, etc., the object being to furnish each of these classes of students with appropriate collateral reading in connection with their regular mathematical courses, and in a more general sense to furnish certain side lights upon the mathematics belonging to these various periods of study. Such lists should be relatively short and, where conditions permit, the books indicated might well be kept in the actual class room where, if properly shelved and labelled, they would at least attract attention and have a general salutary effect.

Besides the lists just mentioned, it has been suggested that a list of reference books be prepared suitable for purchase by a general college library. Such a list would naturally be relatively long, indicating what the college could well endeavor to accumulate in the course of several years. By way of general lists of a shorter nature, it might be proper also for the committee to work up a "five-foot shelf" of mathematics for colleges and possibly another such shelf for high schools.

The preparation of these various lists, if carefully done, will evidently require some time. It is the intention of the committee to proceed in this direction as

fast as possible and eventually make known its findings in the MONTHLY. How far it would be wise to confine the lists to books in the English language is a question upon which the committee would be glad to have the opinion of all interested persons.

2. It has been suggested that the Association, through its library committee, should form a sort of medium of exchange for books and periodicals. For example, one college may have material, such as duplicates of books, which it does not need but which would be of value to some other school or college. In such cases the Association should be in a position to assist the exchange, more especially in expediting and rendering it less artificial than at present. Such exchanges, though already common among colleges and universities for material in general, could doubtless be accomplished, so far as mathematics is concerned, in a much more effective way than at present, and the medium of publicity afforded by the MONTHLY could evidently be used to much advantage in this direction. If this idea proves valuable, the function of the committee would seem to be that of conducting a clearing house for such material, using the columns of the MONTHLY as may seem desirable.

In connection with this matter of exchanges, it seems clear that the Association should keep on file in its library copies of all American journals and periodicals such as are devoted to college or high-school mathematics, and that these should be ready for loaning out to members as desired. For the sake of completeness, the back volumes of such periodicals, whenever procurable, should at once be secured. Eventually, the question of obtaining foreign periodicals will also need consideration. Teachers in the small colleges do not ordinarily have access to foreign periodicals of this kind except as they borrow them from the large universities which in turn prefer not to loan material of this kind. So it seems highly desirable that, as soon as conditions abroad permit, the Association should arrange to have on file in its library a complete set of the French, Italian, German and British journals dealing with collegiate and high-school mathematics, including their history.

3. Mention has just been made of the Association library, but it must be admitted that only the beginnings of such a library are as yet in existence. It will be the evident duty of the committee to do whatever it can to build up such a library. As soon as it begins to take definite shape, rules for its administration will need to be formulated. In this connection, the precedents already established by the library of the American Mathematical Society and which have proved altogether satisfactory there will naturally suggest the course to be followed in the present instance. In particular, authors should be encouraged at all times to present copies of their publications to the library, and individuals generally should feel that whatever aid they can lend to the enterprise will be very gratefully received.

4. Various other suggestions reaching the committee but as yet remaining in a somewhat embryonic state are the following:

(*a*) That the Association through the library committee lend its influence

to the formation of mathematical reading circles throughout the country. No doubt the committee should at least be in position to suggest a suitable group of books for the study of any one topic which it may be desired to read in this way.

(*b*) That means be devised if possible whereby publishing houses will become interested in the general activities of the library and donate books to it by way of advertisement.

In conclusion, it would seem that the phenomenal interest and growth attending the first year of the Association portends a corresponding early and substantial development of its library and of the general library interests of all its members. The committee can but hope to aid in all possible ways to bring this about and at the present time any suggestions to this end beyond those mentioned above would be very gratefully received. These should be sent to the chairman, Professor W. B. Ford, 904 Forest Avenue, Ann Arbor, Mich.

Following the reading of the report by Professor Ford, a number of persons spoke. Professor Huntington emphasized the great value of the contemplated plans of the committee and the usefulness in particular of check lists of books in the libraries of our institutions of learning. Professor Richard Morris said that inspired by the action of the teachers of the public schools of New Jersey, the state librarian at Trenton had expressed his willingness to furnish books called for by the teachers. Professor F. J. Holder told of similar aid afforded by the Carnegie libraries, and Professor J. N. Van der Vries reported the services rendered to institutions of the middle west by the John Crerar Library of Chicago.

In contrast to the selection by Dr. Gronwall of a number of books in foreign languages, Professor R. B. Allen urged that most of the books for students should be in English, inasmuch as those interested in mathematics have not so large an interest or ability in languages. Professor W. R. Ransom suggested that where foreign books are put into such lists, there should be an indication of their size, difficulty, and accessibility. Professor W. A. Hurwitz remarked that while many satisfactory texts in elementary subjects exist in English, there is a dearth of texts in English for intermediate courses, and that, when there are no English books of the desired sort, the presence of foreign books in these lists will arouse publishers to a sense of the desirability of publishing such books.

Professors H. E. Slaught and H. E. Hawkes spoke of mathematical clubs and of the selection of a good list of topics for these clubs to be used in connection with appropriate references to available books. When the latter told of thirty or forty topics which have actually been used at Columbia University during the past few years, he was requested by the meeting to publish in the MONTHLY this list together with the accompanying references to accessible sources. Two or three others spoke in regard to the exchange of books and on reading circles.

It was voted that the report of the Library Committee be placed on file for publication, and that the committee be encouraged to continue in its valuable work.

It was voted that the general program committee for the next meeting of the

Association be asked to make a suitable place on its program for such topics as are of institutional interest, and that, if occasion require any decision by formal vote, an executive session of the institutional delegates be held. The meeting then adjourned to make way for the annual business meeting.

Annual Business Meeting.

The Secretary-Treasurer reported the death during the year of the following ten charter members of the Association:

L. L. Conant, John E. Sinclair Professor of Mathematics, Worcester Polytechnic Institute.
W. C. Esty, Professor of Mathematics, Emeritus, Amherst College.
F. W. Frankland, Consulting Actuary, New York, N. Y.
F. P. Hebblethwaite, former Instructor in Mathematics, Northwestern University.
A. H. Holmes, Lawyer, Brunswick, Me.
Dr. Emory McClintock, Consulting Actuary, Bay Head, N. J.
Mrs. Eva S. Maglott, Professor of Mathematics, Ohio Northern University.
J. C. Rayworth, Assistant Professor of Mathematics, Washington University.
H. A. Sayre, Professor of Mathematics, University of Alabama.
A. G. Smith, Head of the Department of Mathematics, University of Iowa.

The election of officers for the year 1917 was conducted both by mail and in person at this meeting, as provided by the constitution.

The tellers (Professor H. E. Hawkes, G. H. Ling, and W. R. Ransom) appointed by President Hedrick reported the result of the balloting as follows:

For President, Florian Cajori, Colorado College.
For Vice-Presidents, Oswald Veblen, Princeton University, and
D. N. Lehmer, University of California.
For Secretary-Treasurer: W. D. Cairns, Oberlin College.
For additional members of the Executive Council to serve until January, 1920:

E. R. Hedrick, University of Missouri,
D. E. Smith, Columbia University,
R. E. Moritz, University of Washington,
Helen A. Merrill, Wellesley College.

A full description of the interest in this election was published in the January Monthly.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to the date of December 21, 1916. The report was approved subject to an inspection by the auditing committee (Professors R. G. D. Richardson, H. E. Slaught, and A. H. Wilson) appointed by the president. This committee made its inspection later in the day, and formally approved the report. This report is printed in full below.

TREASURER'S REPORT FOR THE YEAR 1916.

RECEIPTS.

Balance from 1915 business		$ 958.72
1916 subscriptions	$ 502.86	
1916 indiv. memberships	3,110.90	
1916 instit. memberships	271.60	
1916 initiation fees	24.00	
Sale copies of MONTHLY	49.15	
Sale reprints	18.77	
Advertising	392.00	
Exchange	2.02	
Interest State Savgs. Bk.	39.71	
Interest Peoples Bk.	13.36	
Total 1916 receipts		4,424.37
Total receipts, 1915–1916		$5,383.09
Balance on 1915–1916 business		$1,671.47
Recd. on 1917–1920 business		1,281.77
Book balance Dec. 21, 1916		$2,953.24

EXPENDITURES.

Publisher's bills		$2,688.19
Paid for reprints		13.74
President's office		126.65
Managing editor's office		108.11
Other editors' postage		16.30
Secretary-Treasurer's office:		
Postage	$229.00	
Bond	5.00	
Desk and office supplies	58.53	
Express, telegrams, freight, etc.	46.72	
Clerical work	239.35	
Printing	106.72	
Cambridge meeting	69.18	
Institutional meeting	4.13	
		758.63
Total expenditures		$3,711.62
Cash on hand		$ 3.00
Checking account		1,176.85
State Savgs. Bk. Co. account		1,260.03
Peoples Bkg. Co. account		513.36
Bank balance Dec. 21, 1916		$2,953.24

Approved by auditing committee,

R. G. D. RICHARDSON,
H. E. SLAUGHT,
A. H. WILSON.

December 29, 1916.

When the accounts were closed on December 21, 1916, for the purposes of the above record, there remained on the total business for the calendar year 1916 the following items:

BILLS RECEIVABLE.

Advertising	$ 86.86
1916 dues unpaid	30.00
Back subscriptions (estim.)	10.00
Due on reprints	1.40
	$128.26

BILLS PAYABLE (all estimated).

Printing December issue	$250.00
2d-class postage Jan.–March	150.00
Printing charter membership list	200.00
Printing New York program	12.00
	$612.00

It will be seen from this report that the former management of the MONTHLY transferred to the Association $958.72. With this in mind the Council through its Committee on Finance has set aside one thousand dollars to be kept as a reserve fund. It may be noted also that the business for the calendar year 1916 alone will thus close with a probable balance a little under two hundred dollars, and that more than $1,200 has already been paid into the treasury on the business of the new year.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

The Council met at nine o'clock Friday morning and held other short meetings between the various sessions of the Association, nine members being present. The principal business transacted is indicated herewith.

(1) The following sixteen institutions, on applications duly certified, were elected to institutional membership, making the total number now 76:

Woman's College of Alabama, Montgomery, Ala.
Trinity College, Hartford, Conn.
Boston University, Boston, Mass.
Mount Holyoke College, South Hadley, Mass.
Worcester Polytechnic Institute, Worcester, Mass.
Michigan Agricultural College, East Lansing, Mich.
Princeton University, Princeton, N. J.
The Polytechnic Institute, Brooklyn, N. Y.
Hamilton College, Clinton, N. Y.
Columbia University, New York, N. Y.
Rochester University, Rochester, N. Y.
Union University, Schenectady, N. Y.
Lafayette College, Easton, Pa.
Lehigh University, South Bethlehem, Pa.
Washington and Jefferson College, Washington, Pa.
Brown University, Providence, R. I.

(2) The following fifteen persons, on applications duly certified, were elected to individual membership, making the total number now 1,064, deducting the number of those who have died during 1916.

J. Q. McNatt, with the Colorado Fuel and Iron Co., Florence, Col.
W. C. Welling, Trinity College, Hartford, Conn.
E. B. Miller, University of Kansas, Lawrence, Kan.
G. A. Osborne, Massachusetts Institute of Technology, Boston, Mass.
C. A. Shook, Graduate School, Harvard University, Cambridge, Mass.
Eleanor C. Doak, Mount Holyoke College, South Hadley, Mass.
Vera L. Wright, University of Minnesota, Minneapolis, Minn.
C. J. Payne, State Normal School, Cape Girardeau, Mo.
P. H. Daus, University of New Mexico, Albuquerque, N. M.
J. B. Rosenbach, Graduate School, University of New Mexico, Albuquerque, N. M.
Vevia Blair, Graduate School, Columbia University, New York, N. Y.
Oscar Hoppe, with the American Circular Loom Co., New York, N. Y.
Louise D. Cummings, Vassar College, Poughkeepsie, N. Y.
E. A. Painter, The Yeates School, Lancaster, Pa.
H. M. Manning, Surgeon, U. S. Public Health Service, Charleston, S. C.

(3) A section of the Association was established for Maryland and the

District of Columbia, with the possible inclusion of Virginia. Professor Abraham Cohen, of Johns Hopkins University, is the secretary.

(4) A committee consisting of Professor Huntington, chairman, Professor Cajori and the secretary-treasurer was appointed with power to determine the time and place of the summer meeting, in conference with a similar committee of the American Mathematical Society.

(5) It was voted to appoint a committee which should in conjunction with a similar committee of the Society consider the question of possible assistance for *Revue Semestrielle* and the *Jahrbuch über die Fortschritte der Mathematik*. The committee was empowered to include also in its investigation other international projects of a kind similar to the two named. Mathematicians the country over are feeling increasingly the deplorable influence of the European war as it affects such indispensable aids as the German and French encyclopædias, the two journals above mentioned, and similar reference books. This action has been taken in order that the two great mathematical organizations of America may consider what contribution they may perhaps make in rendering assistance to these valuable journals of record.

(6) It was voted to hold the next annual meeting in Chicago in conjunction with the Chicago meeting of the American Mathematical Society.

(7) In a session following the election of officers, the Council, in pursuance of its constitutional authority to fill vacancies *ad interim*, filled the vacancy caused by the election of Professor Cajori to the presidency by the appointment of Professor E. V. Huntington, to serve until January, 1918.

(8) The members of the Committee on Publications (H. E. Slaught, managing editor, R. D. Carmichael, and W. H. Bussey) were reappointed for the year 1917.

(9) A Committee on Membership with ex-President Hedrick as chairman was authorized by the Council.

(10) The president-elect was empowered to make the necessary modifications in the existing committees of the Council and to appoint the new committees already authorized. He has accordingly appointed the following:

Committee on Sections: D. E. Smith, Chairman; E. R. Hedrick, M. B. Porter.

Committee on Membership: E. R. Hedrick, Chairman; E. V. Huntington, M. W. Haskell.

W. D. CAIRNS, *Secretary-Treasurer.*

ON THE ORIGIN OF CERTAIN TYPICAL PROBLEMS.[1]

By DAVID EUGENE SMITH.

One thing which impresses the student of mathematical problems is that several which he would naturally classify as purely fictitious and of the nature of pleasing puzzles apparently had their origin in genuine applications of mathematics to questions of real life. Of these I shall mention only four, although the list could be greatly extended.

[1] Extract from a paper on the History of Mathematical Recreations, read before the Mathematical Association of America at Cambridge, Mass., September 1, 1916.

The first of these problems, without which an algebra of to-day might by some be thought to be incomplete, so rooted is it in our traditions, is that of the pipes filling the cistern. No problem has had a longer and more continuous history, and the traveler who is familiar with the Mediterranean lands cannot fail to recognize that here is its probable origin. Not a town of any size that bears the stamp of the Roman power is without its public fountain into which or from which several conduits lead. In the domain of physics, therefore, this would naturally be the most real of all the problems that came within the purview of every man, woman, or child of that civilization. Furthermore, the elementary clepsydra[1] may also have suggested the same line of problems, the principle involved being the same.

The problem in definite form first appears in Heron's Μετρήσεις of about 100 A. D., and although there is some question as to the authorship and date of the work, there is none as to the fact that this style of problem would appeal to such a writer as he. It next appears in the writings of Diophantus, c. 275 A. D.,[2] and among the Greek epigrams attributed to Metrodorus, c. 325 A. D., and soon after this it became common property in the east as well as the west. It is found in the list attributed to Alcuin (c. 825); in the great classic of India, the *Lilāvati* of Bhāskara[3] (c. 1150); in the best-known of all the Arab works on arithmetic, the *Kholāsat-el-hisāb* of Behā-ed-dīn (1547–1622); and in numerous medieval manuscripts. When books began to be printed it was looked upon as one of the stock problems of the race, and many of the early writers gave it a prominent position, among them being men like Petzensteiner (1483), Tonstall (1522), Gemma Frisius (1540), and Robert Recorde (c. 1540).[4]

Such, then, was the origin of what was once a cleverly stated problem of daily life. There is, however, this interesting law of book writers—that most of them will steal from one another without the least scruple if they can thinly veil the theft. This problem, therefore, like dozens of others, went through many metamorphoses, of which I shall mention only a few.

In the fifteenth century, and very likely much earlier, there appeared the variant of a lion, a dog, and a wolf, or other animals, eating a sheep,[5] and this form was even more common in the sixteenth century.[6]

[1] Attributed to Plato but improved by Ctesibus of Alexandria. On the whole subject of clepsydræ see Marquardt, J., *La vie privée des Romains*, French edition, Paris, 1893, p. 458.

[2] In Bachet's edition (the Fermat edition of 1670, p. 271) appears this metrical translation:

Totum implere lacum tubulis è quatuor, uno
Est potis iste die, binis hic & tribus ille,
Quatuor at quartus.
Dic quo spatio simul omnes.

[3] See Taylor's translation, p. 50; Colebrooke translation, p. 42.

[4] In Recorde it appears for the first time in English: "Ther is a cestern with iiij. cocks, conteinyng 72 barrels of water, And if the greatest cocke be opened, the water will auoyde cleane in vj howers," etc. *Ground of Artes*, 1558 edition, folio A, 7 v.

[5] Johann Widman (1489) under the chapter title "Eyn fasz mit 3 zapffen." His form is: "Lew Wolff Hunt Itm̄ des gleichen 1 lew vnd 1 hunt vn̄ 1 wolff diese essen mit einander 1 schaff. Vnd der lew esz das schaff allein in einer stund. Vnd d' wolf in 4 stunden. Vnd der hunt in 6 stunden. Nun ist die frag wan sy dass schaff all 3 mit einād̄er essen / in wie lāger zeit sy das essen." 1509 edition, folio 92; 1519 edition, folio 112.

[6] Thus Cataneo, *Le Pratiche*, 1546; Venice edition of 1567, folio 59v: "Se un Leone mangia

In the sixteenth century we also find in various books the variant of the case of men building a wall or a house, in place of pipes filling a cistern, and this form has survived to the present time. It appeared in Tonstall's exhaustive treatise, *De Arte Supputandi*, in 1522,[1] in Cataneo's well-known work of 1546,[2] and in due time became modified to the form beginning, "If *A* can do a piece of work in 4 days, *B* in 3 days," and so on.

The influence of the wine-drinking countries shows itself in the variant given by that remarkable writer Gemma Frisius (1540),[3] who states that a man can drink a cask of wine in 20 days, but if his wife drinks with him it will take only 14 days, from which it is required to find the time it would take his wife alone.

The influence of a rapidly growing commerce led one of the German writers of 1540 to consider the case of a ship with 3 sails, by the aid of the largest of which a voyage could be made in 2 weeks; with the next in size in 3 weeks, and with the smallest in 4 weeks, it being required to find the time if all three were used, several factors being evidently ignored, such as one sail blanketing the others and the speed not being proportional to the power.[4]

The agricultural interests changed it to a mill with four "Gewercken,"[5] and other interests continued to modify it further until, as is usually the case, the style of problem has tended to fall from its own absurdity. Merely mentioning one of our modern writers who modifies the problem to the case of the pipes of a gasolene tank in a motor car, I may close its varied history by referring to a writer of the early nineteenth century,[6] moved by a bigotry which we would not countenance in academic circles to-day, who proposed to substitute priests praying for souls in purgatory.

Thus we see a recreative problem, starting as an ingenuously worded practical case, becoming fictitious under changed conditions, maintaining itself for two thousand years because of its recreative feature, and almost falling by the wayside because of the absurdities which finally attached to it. It is likely to retain, however, some minor place in our schools because it is not only real within the imagination of pupils, which our technical mechanical problems usually are not, but it is interesting and illustrates a valuable mathematical principle.

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in 2. hore una pecora, & l'Orso la mangia in 3. hore, & il Lεopardo la mangia in 4. hore, dimandasi cominciando a mangiare una pecora tutti e 3. a un tratto in quanto tempo la finirebbono."

This form is also found in J. Albert's work of 1540 (1561 edition, folio Nviii), in Coutereel (1631 edition, p. 352) and in the works of numerous other writers.

[1] With the statement that it is similar to the one about the cistern pipes: "Questio hæc similis est illi de cisterna tres habete fistulas: et simili modo soluenda." Folio f. 1.

[2] See folio 60v of the Venice edition of 1567.

[3] 1563 edition of his arithmetic, folio 38.

[4] "Item / 1 ein Schiff mit 3 Siegeln gehet vom Sund gen Riga / Mit dem grösten allein / in 2 wochen / Mit dem andern / in 3 wochen / Vnnd mit dem kleinsten / in 4 wochen," etc. J. Albert (1540), 1561 edition, folio Nvii.

[5] "Ein Mülmeister hat ein Müle mit vier Gewercken / Mit dem ersten mehlt er in 23 stūden 35 Scheffel / Mit dem andern 39 Scheffel / Mit dem dritten 46 Scheffel / Vnnd mit dem vierten 52 Scheffel," etc. The question then is how long it will take them together to grind 19 Wispel (1 Wispel = 24 Scheffel). *Ibid.*

[6] Hay, *The Beauties of Arithmetic*, 1816, p. 218.

The next problem to which I wish to call your attention has not maintained its place in our books although it has an honorable history of over 2,000 years; it is interesting, it is real within the realm of the pupil's imagination; but it fails for the reason that no principle is involved that is needed in secondary mathematics. The problem is the one commonly known as the Josephsspiel, or the one of the Turks and Christians. It relates that 15 Turks and 15 Christians were on a ship and that half had to be sacrificed; it being necessary to choose the victims by lot, the question is as to how they can be arranged in a circle so that, in counting round, every fifteenth should be a Turk.

It is probable that the problem goes back to the custom of *Decimatio* in the old Roman armies, the selection by lot of every tenth man when a company had been guilty of cowardice, mutiny, or loss of standards in action. Both Livy (ii, 59) and Dionysius (ix, 50) speak of it in the case of the mutinous army of the consul Appius Claudius (B. C. 471), and Dionysius further speaks of it as a general custom. Polybius (vi, 38) says that it was a usual punishment when troops had given way to panic. The custom seems to have died out for a time, for when Crassus resorted to decimation in the war of Spartacus he is described by Plutarch (Crassus, 10) as having revived an ancient punishment. It was extensively used in the civil wars and was retained under the Empire, sometimes as *vicesimatio* (every twentieth man being taken), and sometimes as *centesimatio* (every hundredth man).

Now it is very improbable that those in charge of the selection would fail to have certain favorites, and hence it is natural that there may have grown up a scheme of selection that would save the latter from death. Such customs may depart, but their influence remains in various ways. In the present great war we have frequently read of a regiment being decimated; but how few of us have thought of the origin of the expression.[1]

In its semi-mathematical form it is first referred to in the work of an unknown author, possibly Ambrose of Milan, who wrote, under the nom de plume of Hegesippus, a work *De bello iudaico*.[2] In this work he refers to the fact that Josephus, the author of the well-known history of the wars of the Jews, was saved on the occasion of a choice of this kind.[3] Indeed, Josephus himself refers to the matter of his being saved by lucky chance or by the act of God.[4]

The oldest European trace of the problem, aside from that of Hegesippus, is found in Codex Einsidelensis No. 326, of the beginning of the tenth century. It is also referred to in a manuscript of the eleventh century now in the Munich library and in Codex Bernensis No. 704, of the twelfth century. It is given in the *Ta'hbula* of Rabbi ben Esra (d. 1167) in the twelfth century, and indeed

[1] Lucas, in his *Arithmétique Amusante*, p. 17, also suggests the origin of the problem in the custom of *decimatio*.

[2] Edited by C. F. Weber and J. Caesar, Marburg, 1864. See Ahrens, Math. Unterh. u. Spiele, p. 286.

[3] "Itaque accidit ut interemtis reliquis Iosephus cum altero superesset neci." Quoted from Ahrens, l. c.

[4] Καταλείπεται δὲ οὗτος, εἴτε ὑπὸ τύχης χρὴ λέγειν εἴτε ὑπὸ Θεοῦ προνοίας σὺν ἑτέρῳ.

it is to this writer that Elias Levita, who seems first to have given it in printed form (1518), attributes its authorship.

The problem, as it came to be stated, related that Josephus at the time of the sack of the city of Jotapata by Vespasian, hid himself with forty other Jews in a cellar. It becoming necessary to sacrifice some of the number, a method analogous to the old Roman method of *decimatio* was adopted, but in such way as to preserve himself and a special friend. It is on this account that the Germans still call the problem by the name of Josephsspiel.

Chuquet (1484) mentions the problem, as does at least one other writer of the fifteenth century.[1] When, however, printed works on algebra and higher arithmetic began to appear, it became well known. The fact that such writers as Cardan[2] and Ramus[3] gave it prominence was enough to assure its coming to the attention of scholars.[4]

Like so many curious problems, this one found its way to the Far East, appearing in the Japanese books as relating to a mother-in-law's selection of the children to be disinherited. With characteristic Japanese humor, however, the woman was described as making an error in her calculations so that her own children were disinherited and her step-children received the estate.[5]

The third problem of which I think the origin is worth our attention is the common one of the testament. It relates that a man about to die made a will bequeathing $\frac{1}{3}$ of his estate to his widow in case an expected child was a son, the son to have $\frac{2}{3}$; and $\frac{2}{3}$ to the widow if the child was a daughter, the daughter to have $\frac{1}{3}$. The issue was twins, one a boy and the other a girl, and the question was as to the division of the estate.

The problem in itself is of no particular interest, being legal rather than mathematical; but I mention it because it is a type and is by no means isolated. Under both the Roman and the Oriental influence these inheritance problems played a very important rôle in such parts of analysis as the ancients had developed. In the year 40 B. C. the *lex Falcidia* required at least $\frac{1}{4}$ of an estate to go to the legal heir. If more than $\frac{3}{4}$ was otherwise disposed of, this had to be reduced by the rules of partnership. Problems involving this "Falcidian fourth" were therefore common under the Roman law, just as problems involving the widow's dower right were and are common in the English law and in this country.

The problem as I have stated it appears in the writings of Juventius Celsus, a celebrated jurist of about 75 A. D., who wrote on testamentary law; in those of Salvianus Julianus, a jurist in the reigns of Hadrian (117–138), and Antoninus Pius (138–161), and in those of Cæcilius Africanus (c. 100), celebrated for his knotty legal puzzles.[6]

[1] Anonymous MS. in Munich. See *Bibl. Math.*, 1893, p. 32; M. Curtze, *ibid.*, IX (2), 33; VIII (2), 116; X (2), 29; *Abhandlungen zur Geschichte der Math.*, III, 123.

[2] In his *Arithmetica* of 1539.

[3] In his edition of 1569, p. 125.

[4] It is also in Thierfelder's arithmetic (1587, p. 354), in Wynant van Westen's *Mathemat. Vermaecklyckh* (1644 edn. I, p. 16), in Wilken's arithmetic of 1669 (p. 395), and in many other early works.

[5] See Smith and Mikami, *History of Japanese Mathematics*.

[6] Coutereel (Eversdyck edition of 1658, p. 382) traces the problem back to lib. 28, title 2,

In the Middle Ages it was a favorite conundrum, and in the early printed arithmetics it is often found in a chapter on inheritances which reminds us of the Hindu mathematical collections.[1] It went through the same later development that characterizes most problems and finally fell on account of its very absurdity. That is, Widman (1489) takes the case of triplets, one boy and two girls,[2] and in this he is followed by Albert (1540) and Rudolff (1526).[3] Cardan (1539) complicates it by supposing 4 parts to go to the son and 1 part to the mother, or 1 part to the daughter and 2 parts to the mother, and in some way decides on an 8, 7, 1 division.[4] Texeda (1545) supposes 7 parts to go to the son and 5 to the mother, or 5 to the daughter and 6 to the mother,[5] while other writers of the sixteenth century complicate the problem even more.[6] The final complications of the "swanghere Huysvrouwe" or "donna grauida" are found in some of the Dutch books, and these and the change in ideas of propriety account for the banishment of the problem from books of our day.[7] The most sensible remark about the problem to be found in any of the early books is given in the words of the "Scholer" in Robert Recorde's *Ground of Artes* (c. 1540): "If some cunning lawyers had this matter in scanning, they would determine this testament to be quite voyde, and so the man to die vntestate, because the testament was made vnsufficient."[8]

The fourth problem to whose origin and development I wish to direct special attention is the one of pursuit. It would be difficult to conceive of a problem that would seem more real, since we commonly overtake a friend in walking, or are in turn overtaken. It would therefore seem very certain that this problem is among the ancient ones in what was once looked upon as higher analysis. We have a striking proof that this must be the case in the famous paradox of Achilles and the Tortoise, the history of which has been so carefully and entertainingly worked out by our colleague, Professor Cajori. It is a curious fact, however, that it is not to be found in the Greek collections, although it must also be said that we have not a single work on the Greek *logistice* (λογιστική) extant, so that

law 13 of the *Digest* of Julianus. He gives the usual 4, 2, 1 division as followed by Tartaglia, Rudolff, Forcadel, Ramus, Trenchant, vander Schuere, Mellema, and vander Gucht. Coutereel, however, argues for the 4, 3, 2 division, and in this he has the support of Anth. Smijters. Peletier gives 2, 2, 1, and Chauvet gives 9, 6, 4. Brief historical notes appear in other books, as in the Schonerus edition of Ramus (1586 edition, p. 186).

[1] Thus we have "Ein Testament" (Widman), "Erbteilung vnd vormundschaft" (Riese), "Erf-Deelinghe" (Vander Schuere), and "Erbtheilugs-Rechnung" (Starcken).

[2] Edition of 1558, folio 97. He then divides the property in the proportion 4, 2, 1, 1.

[3] Unger, p. 109.

[4] *Arithmetica*, cap. 66, ex. 87.

[5] Folio Xliij.

[6] Ghaligai (1552, folio 65), Köbel (1518, folio Fij), Riese (*Rechnung nach lenge*, 1550, folios 43, 100), Trenchant (1571, 1578 edition, p. 328), Vander Schuere (1600, folio 96), Peletier (1607 edition, p. 244), Coutereel (1631 edition, p. 358), Starcken (1714 edition, p. 444), Tartaglia (*Tvtte l'opere d'aritmetica*, 1592 edition, II, p. 136).

[7] "Soo ontfangt sy ter tijdt haerder baringhe eenen Sone met een Dochter / ēn een Hermaphroditus, dat is / half Man / half Vrouwe." Vander Schuere, 1600, folio 98. In this case he divides 3175 guldens thus: d. 254, m. 508, s. 1524, h. 889. The same problem appears in Clausberg, *Demonstrative Rechen-Kunst*, 1772.

[8] 1558 edition, folio X 8.

it may have been common without our knowing of the fact. It appears, however, among the *Propositiones ad acuendos juvenes* attributed to Alcuin, in the form of the hound pursuing the hare,[1] and thereafter it was looked upon as one of the stock questions of European mathematics. I have run across it in an Italian manuscript of c. 1440, it is in Petzensteiner's work of 1483,[2] Calandri used it in 1491,[3] Pacioli gives it in his Suma of 1494,[4] and most of the writers of any prominence in the sixteenth century embodied it in their lists.[5]

In those centuries when commercial communication was wholly by means of couriers who traveled regularly from city to city, a custom still determining the name of *correo* for a postman in certain parts of the world, the problem of the hare and hound naturally took on the form of, or perhaps paralleled, the one of the couriers. This problem was not, however, always one of pursuit, since the couriers might be traveling either in the same direction or in opposite directions.[6] This variant of the stock problem is purely Italian, for even the early German writers give it with reference to Italian towns.[7] As a matter of course also, it was varied by substituting ships for couriers,[8] while our modern text-book writers show their lack of originality by merely substituting automobiles for ships.

It was natural to expect that the problem should have a further variant, namely, the one in which the couriers should not start simultaneously. In this form it first appeared in print in Germany in 1483,[9] in Italy in 1484,[10] and in England in 1522.[11]

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[1] "De cursu canis ac fuga leporis."

[2] Folio 54; Unger, p. 106.

[3] "Una lepre e inanzi aun chane 3000 passi et ogni 5 passi delcane sono *p* 8 diquegli della lepre uosapere inquanti passi elcane ara giũto lalepre."

[4] "Vna lepre e dinançe a vn cane passa .60. e per ogni passa .5. che fa el cane la lepre ne fa .7. e finalmente el cane lagiongni [la giongi in the edition of 1523, from la giúgnere, to overtake her] dimando in quanti passa el cane giõgera la lepre." Folio 42v. He says that the problem is not clear because we do not know whether the "passa .60." are leaps of the dog or of the hare, showing that he felt bound to take the stock problem as it stood without improving upon the phraseology. Indeed, we have few such marked examples of plagiarism, in that era of universal literary theft, as Pacioli's Sũma.

[5] Thus Rudolff (*Kunstlich rechnung*, 1526, 1534 edition, folio Nvj); Köbel (*Rechenbuch*, 1531, 1549 edition, folio 88, under the title "Von Wandern über Landt," with a picture in which the hare is quite as large as the hound); Cardan (*Arithmetica*, 1539, cap. 66); Wentzel (1599, p. 51); Ciacchi (*Regole generali d'Abbaco*, Firenze, 1675, p. 130); Coutereel (*Cyffer-Boek*, 1690 edition, p. 584), and many others.

[6] Various types are given in Pacioli's *Sũma* of 1494, folio 39.

[7] Thus Petzensteiner (1483, Folio 53), in his chapter "Von wandern," makes the couriers go to "rum" (Rome), thus: "Es sein zween gesellen die gand gen rum. Eyner get alle tag 6 meyl der ander geth an dem ersten tage 1 meyl an dem andern zwue etc. unde alle tag eyner meyl mer dan vor. Nu wildu wissen in wievil tagen eyner als vil hat gangen als der ander." Günther, *Geschichte*, p. 304; Müller, *Deutsche Blätter*, VI, p. 88.

[8] Thus Calandri (1491) says: "Una naue ua da Pisa a Genoua in 5 di: unaltra naue uiene dageno ua a pisa in 3 di. uo sapere partendosi in nun medesimo tempo quella da Pisa per andare a Genoua et quella da Genoua *p* andare a pisa in quanti di siniscon terrano insieme."

[9] Petzensteiner's arithmetic, printed at Bamberg.

[10] Borghi's arithmetic.

[11] Tonstall's *De arte supputandi*, folio 4, "Cvrsor ab Eboraco Londinvm proficiscens," etc. See also Cardan (*Arithmetica*, 1539, cap. 66, with various types); Ghaligai (1521, 1552 edi-

The invention of clocks with minute hands as well as hour hands gave the next variant, as to when both hands would be together—a relatively modern form of the question, as is also the astronomical problem of the occurrence of the new moon. The latest form, however, has to do with the practical question of a railway timetable, but here graphic methods naturally take the place of analysis so that of all the variants those of the couriers and the clock hands seem to be the only ones that will survive. Neither is valuable *per se*, but each is interesting, each is real within the range of easy imagination, and each involves a valuable mathematical principle—a fairly refined idea of function, and so it is probable that each will persist in spite of the present transitory period of the attempted debasement of elementary mathematics.

AN INVERSION OF THE COMPLETE QUADRILATERAL.

By J. W. CLAWSON, Ursinus College.

It is the purpose of this paper to point out an interesting example of the method of inversion. If a complete quadrilateral with some of its related lines and circles be inverted with respect to the quadrilateral's Wallace point, a new complete quadrilateral with some of its related circles and lines results. It is remarkable that this new quadrilateral is inversely similar to the original one, as will appear from II below.

Four straight lines (Fig. 1) *ARB, BCP, CQA, PQR* form a complete quadrilateral with *A, P; B, Q; C, R* for opposite vertices. The lines taken three by three also determine four triangles *ABC, AQR, BRP, CPQ*. It is well known that the circles circumscribing* these triangles are concurrent at a point *O*, the *Wallace point* of the quadrilateral,[1] that the circumcenters of the triangles are concyclic on *l*, the *circumcentric circle*,[2] that the orthocenters of the triangles are

tion, folio 64); Albert (1540, 1561 edition, folio Pi); Baker (1568, 1580 edition, folio 36); Coutereel (1631 edition, p. 371, and Eversdyck edition of 1658, p. 403); Trenchant (1566, 1578 edition, p. 280); Wentsell (1599, p. 51); Peletier (1549, 1607 edition, p. 290), Vander Schuere (1600, folio 179); Schonerus (notes on Ramus, 1586 edition, p. 174), and many others. Köbel's rather quaint German is interesting: "Zwen Burger vsz Oppenheym / einer Sō Heynrich / der ander Contz vō Treber gnant / wolten mit einander gen Rom gehn / vñ Heinrich was alt / vñ mocht einn tag nit mehr dañ zehen meiln gehn / Aber Contz vō Treber was jung vnnd starck / der mocht einen tag 13. meilen gehn / Deszhalben gieng Son Heynrich neun tag eh ausz Oppenheym dann Contz von Treber / Also war Son Heynrich Contzen 90. meilen furgangen / eh Contz angehaben hat auszugehn.

"Nun ist die frag / inn wie vil tagen Contz von Treber / Son Heynrichen übergangen / vnnd die zwen zusamen kommen seind." See also his *Zwey rechenbüchlin*, Frankfort, 1537 edition, folio 84.

* This figure is slightly distorted. The circles should join exactly through the points *R* and *Q*.

[1] "Scoticus," Leybourn's *Mathematical Repository*, 1804, Vol. I, p. 170. Mackay, *Proc. Edin. Math. Soc.*, Vol. IX.

[2] Davies, *Math. Repos.*, 1835, Vol. VI, Question 555 answered.

[1, 2] Steiner, Gergonne's *Annales*, 1828, Vol. XVIII, pp. 302, 303, 1°, 2°, 3°, 4°.

collinear on o, the *orthocentric line*,[3] and that the feet of the perpendiculars from O on the four lines are collinear on p, the *pedal* line[4] of the quadrilateral.

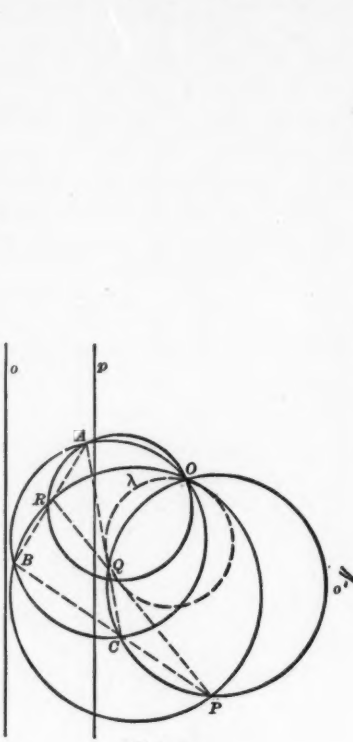


FIG. 1.

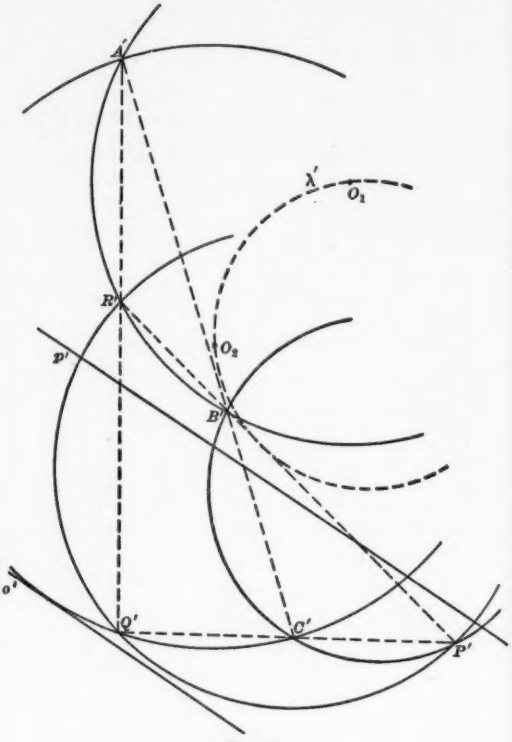


FIG. 2.

Invert the figure with O as center of inversion and with any radius. The full lines of Fig. 1 invert into broken lines of Fig. 2 and vice versa.

I. The circles ABC, AQR, BRP, CPQ of Fig. 1 invert into four straight lines of Fig. 2, $A'B'C'$, $A'R'Q'$, $P'B'R'$, $P'Q'C'$, which form a new quadrilateral with P', A'; Q', B'; R', C' for opposite vertices. The straight lines ARB, BCP, AQC, PQR invert into the four circles, concurrent at O' (not shown on this figure), which circumscribe the triangles $A'R'B'$, $B'C'P'$, $C'A'Q'$, $P'Q'R'$.[5]

II. The new quadrilateral is inversely similar to the old, the center and axis of similitude being respectively O and the bisector of the angle AOP. (This line also bisects angles BOQ and COR.)

For angle A is equal to (or supplementary to) the angle between the lines joining A' to the centers of the circles $A'R'B'$ and $A'C'Q'$. Denote these centers

[1, 2, 4] CASEY, *Sequel to Euclid*, ed. of 1886, pp. 35, 36.

[5] See MCCLELLAND, *Geometry of the Circle*, p. 256, Ex. 12.

by O_1 and O_2. Then $\angle O_1A'O_2 = \angle O_1O'O_2 = \angle O_1O'A' + \angle O_2O'A' = \angle O'B'A' - \angle O'C'A' = \angle C'O'B' = \angle P'$. Thus $\angle A = \angle P'$. Similarly any angle in the figure $APBQCRO$ can be proved equal to the corresponding angle in the figure $P'A'Q'B'R'C'O'$.

III. The orthocentric line of either quadrilateral inverts into the circumcentric circle of the other.

For the pedal line, p, passes through the intersection of BCP and a line through O perpendicular to that line. Hence its inverse is a circle passing through the intersection of the circle $O'B'C'P'$ and a straight line through O' orthogonal to that circle. Hence the inverse of p is a circle passing through the opposite extremities of the diameters through O' of the circles $A'R'B'$, $B'C'P'$, $C'A'Q'$, $P'Q'R'$. Hence the inverse of the orthocentric line, o, which is parallel to p and twice as far from O,[6] is a circle touching the inverse of p at O' and with a radius half as large; that is, it is the circumcentric circle.[7]

IV. Many properties of a complete quadrilateral lead, on inversion, to new properties of this figure. However the results are often complicated and of little interest. One example will be given.

The theorem: "The circles on the three diagonals of the quadrilateral as diameters are coaxial, the radical axis being the orthocentric line,"[8] inverts into:

"The three circles, (1) through A, P orthogonal to the circle OAP, (2) through B, Q orthogonal to OBQ, (3) through C, R orthogonal to OCR, are coaxial; and the circumcentric circle belongs to the same coaxial system."

A PROBLEM IN PROBABILITY.[1]

By C. S. JACKSON, R. M. Academy, Woolwich, England.

1. A problem first proposed by De Moivre and extended by Simpson was thrown into the following form by Laplace: If the numerical result of a single trial is equally likely to have any value between 0 and b, the chance that after n trials the sum of the results obtained shall be less than a is

$$\text{(i)} \qquad \frac{1}{b^n n!}\{a^n - n_1(a-b)^n + n_2(a-2b)^n \cdots\},$$

n_r denoting $n!/[r!(n-r)!]$ and the series being continued as long as $a - rb$ is positive.[2] In the following note an alternative mode of investigating (i) is used, which is intended to illustrate how each term of the formula arises.

2. Let $x_1 \cdots x_n$ be n positive items, each equally likely to have any value

[6] STEINER, *loc. cit.* DAVIES, *loc. cit.* CASEY, *loc. cit.*

[7] Cf. MCCLELLAND, *loc. cit.*

[8] DURELL, *Plane Geometry for Advanced Students*, Part I, Theorem 86, p. 188.

[1] The proof sheets of this article never reached us from the author, having probably been lost in ocean transit. EDITORS.

[2] See TODHUNTER, *History, etc., of Probability*, pp. 84, 208, 542.

between 0 and a, where a lies between the values rb and $(r+1)b$. The chance k_0 that their sum s is less than a is

$$a^{-n}\int_0^a dx_1\int_0^{a-\xi_1} dx_2\int_0^{a-\xi_2} dx_3\cdots\int_0^{a-\xi_{n-1}} dx_n,$$

where $\xi_r = x_1 + x_2 \cdots + x_r$.

This is a well-known integration, or may be worked out by putting

$$z_1 = a - \xi_1, \qquad z_2 = a - \xi_2, \qquad \cdots, \qquad z_n = a - \xi_n,$$

when it becomes

$$a^{-n}\int_0^a dx_1\int_0^{z_1} dz_2\cdots\int_0^{z_{n-1}} dz_n = a^{-n}\frac{a^n}{n!} = \frac{1}{n!}.$$

Again, the chance that $s < a$ and, at any rate, each of m specified items, say $x_1 \cdots x_m$, is greater than b, whatever the others may be, is

$$a^{-n}\int_b^{a-(m-1)b} dx_1\cdots\int_b^{a-(m-r-1)b-\xi_r} dx_{r+1}\cdots\int_b^{a-\xi_{m-1}} dx_m\cdots\int_0^{a-\xi_s} dx_{s+1}\cdots\int_0^{a-\xi_{n-1}} dx_n,$$

the limits being obtained by noticing that $x_1 > b$ and $x_1 < a - (m-1)b$, because at least $(m-1)b$ must be left to provide for $x_2 \cdots x_m$ each exceeding b. We may put for $\lambda > m$

$$z_\lambda = a - \xi_\lambda, \qquad \text{so that} \qquad \int_0^{a-\xi_{\lambda-1}} dx_\lambda = \int_0^{z_{\lambda-1}} dz_\lambda;$$

while for $\lambda \leqq m$ we put

$$z_\lambda = a - (m-\lambda)b - \xi_\lambda, \qquad \text{so that} \qquad \int_b^{a-(m-\lambda)b-\xi_{\lambda-1}} dx_\lambda = \int_0^{z_{\lambda-1}} dz_\lambda;$$

and then the integral becomes

$$a^{-n}\int_0^{a-mb} dz_1\int_0^{z_1} dz_2\cdots\int_0^{z_{n-1}} dz_n = \frac{(a-mb)^n}{a^n n!}.$$

3. The m specified items might be chosen in n_m ways, whence we would write

$$k_m = \frac{(a-mb)^n}{a^n m!(n-m)!},$$

and proceed to analyze k_m into components according to the exact number of items which exceed b.

If u_0 = the chance that $s < a$, when *none* of the items $x_1 \cdots x_n$ exceeds b,
u_1 = the chance that $s < a$, when *one* of the items $x_1 \cdots x_n$ exceeds b,
. . . . . .
u_s = the chance that $s < a$, when *exactly* s of the items $x_1 \cdots x_n$ exceeds b,

and so on (the set ending with u_r, for it is impossible for more than r items to exceed b), then

(ii) $$k_m = u_m + (m+1)_m u_{m+1} \cdots + r_m u_r.$$

The cases which give rise to u_{m+l} are each counted $(m+l)_m$ times in k_m. Putting $m = 0, 1, \cdots r$ in turn in (ii) we obtain

$$k_0 = u_0 + u_1 + \cdots + u_r. \tag{0}$$

$$k_1 = u_1 + 2_1 u_2 + \cdots + r_1 u_r. \tag{1}$$

$$k_m = u_m + (m+1)_m u_{(m+1)} + \cdots + r_m u_r. \tag{m}$$

$$k_{(m+l)} = u_{(m+l)} + \cdots + r_{(m+l)} u_r. \tag{m+l}$$

$$k_r = r_r u_r. \tag{r}$$

To solve these equations, multiply equation m by 1, $\cdots$, equation $(m+l)$ by $(-1)^l(m+l)_m$, $\cdots$, and add.

The coefficient of u_{m+l} in the result is

$$(m+l)_m - (m+l)_{m+1}(m+1)_m + \cdots + (-1)^s(m+l)_{(m+s)}(m+s)_m \cdots + (-1)^l(m+l)_m,$$

which is

$$(m+l)_m\{1 - l_1 + \cdots + (-1)^s l_s \cdots + (-1)^l\} = (m+l)_m(1-1)^l = 0.$$

Thus,

(iii) $$u_m = k_m - (m+1)_m k_{m+1} + (m+2)_m k_{m+2} \cdots + (-1)^{r-m} r_m k_m, \cdots$$

and, in particular,

$$u_0 = k_0 - k_1 + k_2 \cdots + (-1)^r k_r,$$

where, as already shown,

$$k_m = \frac{(a-mb)^n}{a^n m!(n-m)!}.$$

4. Now the probability being u_0 that $s < a$ and that each of the items $x_1 \cdots x_n$ less than b, and the *a priori* probability being $(b/a)^n$ that $x_1 \cdots x_n$ shall each be less than b, then the probability that, when $x_1 \cdots x_n$ are each given less than b, their sum s shall be less than a is

$$\left(\frac{a}{b}\right)^n \times u_0$$

or

(iv) $$\frac{1}{b^n n!}\{a^n - n_1(a-b)^n + n_2(a-2b)^n \cdots\}$$

5. Again, from (iii),

$$u_m = \frac{(a - mb)^n}{a^n \cdot m!(n-m)!} - (m+1)_m \frac{(a - mb - b)^n}{a^n (m+1)!(n-m-1)!} + \cdots$$
$$+ (-1)^s (m+s)_m \frac{(a - mb - sb)^n}{a^n (m+s)!(n-m-s)!} \cdots,$$

and the *a priori* probability being $n_m[(a-b)^m b^{n-m}/a^n]$ that exactly m items exceed b, the chance that, when m items exceed b, their sum s shall be less than a is

$$\text{(v)} \quad \frac{1}{n!(a-b)^m b^{n-m}} \{(a - mb)^n - (n-m)(a - mb - b)^n \cdots$$
$$+ (-1)^s (n-m)_s (a - mb - sb)^n \cdots\}.$$

This last result (v) is the chance that if, out of n positive items, m are equally likely to have any value between b and a, and the remainder to have any value less than b, then their sum shall be less than a.

6. The Hon. R. J. Strutt gave an interesting application of formula iv in the *Philosophical Magazine*, 6 series, Vol. 1, p. 311. The sum of the numerical departures from integral values of nine well-determined atomic weights is .809. If we suppose that an individual departure is equally likely to have any value between 0 and .5, the chance of the sum of nine departures being less than .809 proves to be .001159. The smallness of this value, he infers, gives some support to the well-known hypothesis that the atomic weights should be integers.

A SIMPLE GEOMETRICAL PARADOX.

PROPOSED BY J. L. COOLIDGE, Harvard University.

Suppose that we have an algebraic surface

$$x = \frac{f_1(u, v, w)}{f(u, v, w)}, \qquad y = \frac{f_2(u, v, w)}{f(u, v, w)}, \qquad z = \frac{f_3(u, v, w)}{f(u, v, w)},$$
$$F(u, v, w) = 0.$$

We shall assume that this surface has no singular curve, an assumption which still leaves us in what we may call the *general* case, since the discriminant of a polynomial in three variables does not vanish identically. Let us cut this surface by an arbitrary plane which does not pass through any isolated singularity which the surface may possess, a *general* plane we might say. The coördinates of the points of the curve of intersection are algebraic functions of a single parameter, and the same is true of the sine of the angle which the given plane makes with the tangent plane to the surface at the points of the curve.

Suppose first, that this algebraic function is not a constant. It must, then,

have one or more zeros. At such a point, the sine of the angle, and hence the angle itself, must be zero and the surface touches the plane. This, however, is absurd; no surface can touch every plane not through an isolated singular point. Let us, then, suppose that the algebraic function is a constant. The surface will then meet every plane at a constant angle, every plane through a normal will cut it everywhere orthogonally. Hence each two normals are coplanar, the normals are all concurrent or parallel, and the surface is either a sphere or a plane.

CONCLUSION. *Every algebraic surface which has no singular curve is a sphere or plane.*

ORGANIZATION OF A MINNESOTA SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

On November 8, 1916, a committee consisting of Messrs. G. N. Bauer, R. M. Barton and G. W. Hartwell, representing the Minnesota members of the Mathematical Association of America, sent out a circular letter over the state inviting all teachers of college mathematics to meet at the University of Minnesota on Friday, December 1, 1916, for the purpose of considering the possibility and wisdom of organizing a Minnesota section of the Mathematical Association of America.

Twenty-three teachers attended this meeting sixteen of whom were already, or have since become, members of the Association. These members are as follows: Sister M. Magna, of St. Benedict's College, St. Joseph; Miss E. G. Berger, of St. Catherine's College, St. Paul; Wm. C. Etzel, College of St. Thomas, St. Paul; Paul E. Kretzmann, Concordia College, St. Paul; L. E. Lunn, Supt. of Schools, Heron Lake; Thos. C. Wollan, Fergus Falls; G. N. Bauer, C. McCormick, Wm. O. Beal, H. H. Dalaker, A. L. Underhill, Vera L. Wright, H. L. Slobin, W. H. Bussey, R. M. Barton, and W. D. Reeve, of the University of Minnesota.

In the morning session Dr. Bauer presented the report of the committee in which he discussed some of the problems confronting college teachers in Minnesota. It was then decided by unanimous vote to form a Minnesota Section of the Mathematical Association of America and a committee was chosen to nominate officers for the ensuing year. The meeting then adjourned to a luncheon arranged in Alice Shevlin Hall at the University.

At the beginning of the afternoon session the nominating committee recommended Dr. G. N. Bauer for President, W. D. Reeve for Secretary-Treasurer and Dr. C. N. Gingrich as the third member of an executive committee. In addition the committee selected J. S. Mikesh and Miss E. G. Berger to act with the executive committee as a committee on policy for the section. The recommendations of the nominating committee were accepted.

It was further agreed to hold two meetings of the section each year, one in the spring and one in the autumn, the time to be set definitely by the executive committee.

The following program was then given: "Cultural value of college mathematics," J. S. Mikesh; "Report of current research in transcendental curves and numbers," Dr. H. L. Slobin; "Thoughts on a natural number system," L. E. Lunn, Heron Lake; "A solution of the differential equation $dy/dt + (\alpha + \beta \cos t)y = \rho \cos t$," W. O. Beal; "Unification of mathematics in the high school and college," W. D. Reeve.

W. D. REEVE, *Secretary.*

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

NEW BOOKS RECEIVED.

ELEMENTS OF ANALYTIC GEOMETRY. By Alexander Ziwet and Louis Allen Hopkins. The Macmillan Company, New York, 1916. viii + 272 pages. $1.60.

FIRST YEAR MATHEMATICS. By George W. Evans and John A. Marsh. Charles E. Merrill Co., New York, 1916. 253 pages. $0.90.

PROJECTIVE ORNAMENT. By Claude Bragdon. The Manas Press, Rochester, N. Y., 1915. 79 pages. $1.50.

A PRIMER OF HIGHER SPACE. By Claude Bragdon. The Manas Press, Rochester, N. Y., 1913. 79 pages. $1.25.

FOUR DIMENSIONAL VISTAS. By Claude Bragdon. Alfred A. Knopf, New York, 1916. 134 pages. $1.25.

A SHORT COURSE IN ELEMENTARY MECHANICS FOR ENGINEERS. By Clifford Newton Mills. D. Van Nostrand Co., New York, 1916. xi + 127 pages. $1.00.

INTRODUCTION TO MATHEMATICS. Junior High School Series. By Robert L. Short and William H. Elson. D. C. Heath and Co., Boston, 1916. vii + 200 pages. $1.00.

NEW PLANE AND SOLID GEOMETRY. By Edward Rutledge Robbins. American Book Company, New York, 1916. viii + 460 pages.

Differential Calculus. By H. B. PHILLIPS. John Wiley and Sons, New York, 1916. v + 162 pages.

This small compact volume contains a brief course on differential calculus in 139 pages followed by 14 pages of supplementary exercises and 9 pages devoted to answers to problems and an index. It contains all that can be covered in a semester course of 3 hours a week. It may even be made to serve as the basis of a four- or five-hour course. The following quotation from the preface tells what was the author's idea in writing the book: "In this text on differential calculus I have continued the plan adopted for my Analytic Geometry, wherein a few central methods are expounded and applied to a large variety of examples to the end that the student may learn principles and gain power. In this way the differential calculus makes only a brief text suitable for a term's work and leaves for the integral calculus, which in many respects is far more important, a greater proportion of time than is ordinarily devoted to it."

The reviewer finds the following things about the book worthy of comment: (1) In the "Introduction" (Chapter I), along with the usual definition of infinitesimals, is given the idea of the *order of infinitesimals.* (2) "Derivatives and Differentials" are taken up side by side in Chapter II. (3) Applications of

differential calculus to "Rates" (Chapter IV) and "Maxima and Minima" (Chapter V) are given right after "Differentiation of Algebraic Functions" (Chapter III). "Transcendental Functions" are not introduced until Chapter VI. (4) Maxima and Minima are treated without the use of second derivatives. (5) The number e is introduced in Chapter VI as the number satisfying the relation

$$\lim_{x=0} \frac{e^x - 1}{x} = 1.$$

and the approximate value of e is not computed until Chapter X on "Series and Approximations" is reached. (6) Chapter VII contains "Geometrical Applications" to both plane and solid geometry and they are so arranged that the latter may be omitted if it is desired. (7) There are numerous applications to the simpler problems of mechanics in the plane and in space.

The chapter headings not already mentioned are Chapter VIII, "Velocity and Acceleration in a Curved Path," Chapter IX, "Rolle's Theorem and Indeterminate Forms," and Chapter XI, "Partial Differentiation."

The supplementary exercises at the end of the book (pages 140–153) are to be used as "material for review and to provide problems for which answers are not given.

The book seems to be one which will teach the student not only the mechanical part of differential calculus but also the true value of the processes involved.

A. L. UNDERHILL.

UNIVERSITY OF MINNESOTA.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

475. Proposed by E. B. ESCOTT, Kansas City, Mo.

A man makes a contract to purchase a house, making a cash payment down and agreeing to make monthly payments of a dollars, interest being charged at six per cent., the balance of the monthly payments being credited on the principal. Find a formula for M_n, the balance due after n payments.

476. Proposed by W. HAROLD WILSON, University of Illinois.

Prove that, if $x_h \neq x_j$, $h, j = 1, 2, 3 \cdots n$, $h \neq j$, then

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{h=1}^{n} (x_i - x_h)'} = 1,$$

where the prime indicates the omission of zero factors in the denominators.

GEOMETRY.

508. Proposed by J. E. ROWE, State College, Penn.

The trilinear coördinates of the vertices of the Brocard triangle are $(s_3^3, s_1s_2s_3, s_1^3)$, $(s_2^3, s_1^3, s_1s_2s_3)$, and $(s_1s_2s_3, s_3^3, s_2^3)$, where s_i $(i = 1, 2, 3)$ are the sines of the angles of the funda-

mental triangle. Show that the Brocard triangle and the fundamental triangle are in perspective, and that the trilinear coördinates of the center of perspectivity are s_i^{-3} $(i = 1, 2, 3)$ instead of s_i^3 which are incorrectly given in Clebsch's *Vorlesungen über Geometrie*, p. 323.

509. Proposed by NORMAN ANNING, Chilliwack, B. C.

A picture whose coördinates are (0, 0), (50, 0), (50, 50), and (0, 50) is repeated on a smaller scale as part of itself with the coördinates (7, 0), (31, 7), (24, 31), (0, 24). Locate the vanishing point.

CALCULUS.

423. Proposed by J. B. REYNOLDS, Lehigh University.

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the x-axis is the two-arched epicycloid.

424. Proposed by OSCAR S. ADAMS, Washington, D. C.

What is the value of

$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}?$$

MECHANICS.

340. Proposed by PAUL CAPRON, U. S. Naval Academy.

A rigid straight line l passes through a fixed point O, but is otherwise free to move in a plane. If C is the instantaneous center of rotation for l, prove that CO is always perpendicular to l and that, if (O being used as pole) $\rho = f(\theta)$ represents the locus of any point P on l, OC is always equal to $(d/d\theta)f(\theta)$.

341. Proposed by PAUL CAPRON, U. S. Naval Academy.

A pole l feet long, with one end on the ground, touches the top of a wall a feet high and slides in a vertical plane perpendicular to the wall. Show that its instantaneous center of rotation is at the intersection of the vertical where it touches the ground with the perpendicular to its axis where it touches the wall, and that the locus of this center is a parabola having the latus rectum a.

NUMBER THEORY.

259. Proposed by E. E. WHITFORD, College of the City of New York.

If p is relatively prime to 10, and if any multiple of p consisting of n digits has its digits permuted cyclically, the number thus formed is also a multiple of p; the number n to be determined by the congruence $10^n \equiv 1 \pmod{p}$. For example, 481, 814, and 148 are each multiples of 37.

260. Proposed by ALBERT A. BENNETT, University of Texas.

Let $\binom{n}{r}$ denote as usual the binomial coefficient $n!/[r!(n-r)!]$, where $\binom{n}{0} = 1$, but where n, r, $(n-r)$ are always to be supposed to be positive integers or zero. Let us define $k_i(m, n)$ as $\sum_j \binom{m-i+j}{i-j}\binom{n-j}{j}$. Prove that the following recursion formulas are consistent:

$$\sum_i (-1)^i k_i(m, n) C_{m+n-i} = \binom{m+n}{m}$$

and determine $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1{,}430$, etc. Prove also that these quantities satisfy the following relations, as well:

$$\sum_i (-1)^i C_{m-n-i}\binom{m-i}{i} = 0 \text{ for each } n \text{ where } 2n \leq m.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

462. Proposed by H. S. UHLER, Yale University.

Show how to transform A into S, where these symbols denote the equivalent formulæ for the general case of Calculus Problem No. 363, pages 52 and 54 in the February, 1916, MONTHLY:

$$A \equiv 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R\sin\pi/n}{\sqrt{R^2-a^2}}\right) + 2anR\sin^{-1}\left[\frac{2a(\tan\pi/n)(\sqrt{R^2-a^2\sec^2\pi/n})}{R^2-a^2}\right],$$

$$S \equiv 4nR\left\{a\sin^{-1}\left(\tan\frac{\pi}{n}\cdot\frac{a}{\sqrt{R^2-a^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-a^2\sec^2\pi/n}}{\sqrt{R^2-a^2}}\right]\right\}.$$

SOLUTION BY GEO. W. HARTWELL, Hamline University.

$$A \equiv 4nR\left\{R\left[\frac{\pi}{n} - \sin^{-1}\left(\frac{R\sin\pi/n}{\sqrt{R^2-a^2}}\right)\right] + \tfrac{1}{2}a\sin^{-1}\left[\frac{2a(\tan\pi/n)(\sqrt{R^2-a^2\sec^2\pi/n})}{R^2-a^2}\right]\right\}$$

$$\equiv 4nR(a\gamma + R\beta).$$

Let

$$\sin^{-1}\left(\frac{R\sin\pi/n}{\sqrt{R^2-a^2}}\right) = \alpha \quad \text{and} \quad \frac{\pi}{n} - \alpha = \beta.$$

Then,

$$\sin\beta = \sin\frac{\pi}{n}\cos\alpha - \cos\frac{\pi}{n}\sin\alpha.$$

$$\cos\alpha = \sqrt{1 - \frac{R^2\sin^2\pi/n}{R^2-a^2}} = \sqrt{\frac{R^2\cos^2\pi/n - a^2}{R^2-a^2}} = \frac{\cos\pi/n\sqrt{R^2-a^2\sec^2\pi/n}}{\sqrt{R^2-a^2}}.$$

Hence,

$$\sin\beta = \sin\frac{\pi}{n}\cos\frac{\pi}{n}\left[\frac{\sqrt{R^2-a^2\sec^2\pi/n}-R}{\sqrt{R^2-a^2}}\right],$$

or

$$\beta = -\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-a^2\sec^2\pi/n}}{\sqrt{R^2-a^2}}\right].$$

Let

$$\sin^{-1}\frac{2a(\tan\pi/n)(\sqrt{R^2-a^2\sec^2\pi/n})}{R^2-a^2} = \gamma.$$

Then,

$$\sin\tfrac{1}{2}\gamma = \sqrt{\tfrac{1}{2}(1-\cos\gamma)} = \sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{4a^2\tan^2\pi/n\,(R^2-a^2\sec^2\pi/n)}{(R^2-a^2)^2}}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{R^4-2a^2R^2(1+2\tan^2\pi/n)+a^4(1+4\sec^2\pi/n\tan^2\pi/n)}}{R^2-a^2}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1-\frac{R^2-a^2(1+2\tan^2\pi/n)}{R^2-a^2}\right)} = \frac{a\tan\pi/n}{\sqrt{R^2-a^2}};$$

hence,

$$\tfrac{1}{2}\gamma = \sin^{-1}\frac{a\tan\pi/n}{\sqrt{R^2-a^2}}.$$

Making these substitutions, we have $A \equiv S$.

Also solved by the PROPOSER.

463. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the sum of the series

$$\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n},$$

where $\binom{n}{r}$ denotes the coefficient of x^r in the expansion of $(1+x)^n$.

SOLUTION BY EDWIN R. SMITH, State College, Pa.

Consider the expansion of $1/(1-x-2x^2)$ into a series of ascending powers of x. First,

$$\frac{1}{1-x-2x^2} = \frac{1}{1-(x+2x^2)} = 1 + (x+2x^2) + (x+2x^2)^2 + \cdots$$

$$= 1 + \binom{1}{0}x + \left[\binom{2}{0} + \binom{1}{1}2\right]x^2 + \left[\binom{3}{0} + \binom{2}{1}2\right]x^3 + \cdots$$

$$+ \left[\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n}\right]x^{2n} + \cdots.$$

A second expansion can be obtained as follows:

$$\frac{1}{1-x-2x^2} = \frac{1}{1+x}\cdot\frac{1}{1-2x}$$

$$= (1 - x + x^2 - x^3 + \cdots)(1 + 2x + 2^2x^2 + 2^3x^3 + \cdots)$$

$$= 1 - (1-2)x + (1 - 2 + 2^2)x^2 - (1 - 2 + 2^2 - 2^3)x^3 + \cdots$$

$$+ (1 - 2 + 2^2 - 2^3 + \cdots + 2^{2n})x^{2n} - \cdots$$

$$= 1 - \frac{1-2^2}{1+2}x + \frac{1+2^3}{1+2}x^2 - \cdots + \frac{1+2^{2n+1}}{1+2}x^{2n} - \cdots.$$

Comparing in the two series the expressions for the coefficient of x^{2n} there is obtained

$$\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n} = \frac{1+2^{2n+1}}{1+2} = \tfrac{1}{3}(2^{2n+1}+1),$$

which is the required sum.

Also solved by A. M. KENYON, N. P. PANDYA, and the PROPOSER.

464. Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Find the general term and the sum of n terms of the series 1, 4, 15, 56, $\cdots$, where

$$U_n = 4U_{n-1} - U_{n-2}.$$

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Changing the notation so that for U_n we may write U_{n+2}, we have

$$U_{n+2} - 4U_{n+1} + U_n = 0,$$

an equation in finite differences. Integrating, noticing that the roots of

$$m^2 - 4m + 1 = 0,$$

are

$$m_1 = 2 + \sqrt{3},$$

$$m_2 = 2 - \sqrt{3},$$

$$U_n = C_1(2+\sqrt{3})^n + C_2(2-\sqrt{3})^n. \tag{1}$$

When $n = 1$, $U_1 = 1$, and when $n = 2$, $U_2 = 4$; then (5) gives for determining C_1 and C_2,

$$C_1(2 + \sqrt{3}) + C_2(2 - \sqrt{3}) = 1,$$

$$C_1(2 + \sqrt{3})^2 + C_2(2 - \sqrt{3})^2 = 4,$$

giving $C_1 = \frac{1}{6}\sqrt{3}$ and $C_2 = -\frac{1}{6}\sqrt{3}$.

(1) then becomes $$U_n = \tfrac{1}{6}\sqrt{3}\{(2 + \sqrt{3})^n - (2 - \sqrt{3})^n\},$$

the general term. The usual theory for the sum of n terms gives

$$S_n = C + C_1 m_1 \frac{{m_1}^n}{m_1 - 1} + C_2 m_2 \frac{{m_2}^n}{m_2 - 1}.$$

Substituting the values of m_1, m_2, C_1, C_2, we have

$$S_n = C + \tfrac{1}{6}\sqrt{3}\left\{\frac{(2 + \sqrt{3})^{n+1}}{\sqrt{3} + 1} + \frac{(2 - \sqrt{3})^{n+1}}{\sqrt{3} - 1}\right\}. \tag{2}$$

When $n = 1$, $S_n = 1$, and (2) gives $C = -\frac{1}{2}$, and this in (2) gives the required sum.

Also solved by AMELIA BENSON, G. W. HARTWELL, E. B. ESCOTT, A. M. HARDING, N. P. PANDYA, and O. S. ADAMS.

GEOMETRY.

489. Proposed by NATHAN ALTSHILLER, The University of Oklahoma.

The parallels to the asymptotes a, b of a given hyperbola, drawn from a variable point of the curve, meet a and b in P, Q respectively. The line PQ envelops an hyperbola whose asymptotes are a and b.

I. SOLUTION BY E. J. OGLESBY, Williamsburg, Virginia.

Take the asymptotes a, b as the axes of coördinates. Then the equation of the hyperbola may be taken as $xy = c^2$ and the coördinates of the variable point on the hyperbola as $(ct, c/t)$ in terms of the parameter t. P is the point $(ct, 0)$, and Q is $(0, c/t)$.

The equation of PQ may be written

$$t^2 y - ct + x = 0. \tag{1}$$

We find the envelope of (1) by applying the condition that this equation shall have equal roots in the parameter t.

Hence, we have

$$(-c)^2 - 4yx = 0, \quad \text{or} \quad xy = c^2/4,$$

which is an hyperbola having a and b as asymptotes.

II. SOLUTION BY THE PROPOSER.

The tangents a, b to a given conic at the points A, B, are met by the lines BM, AM joining A and B to a variable point M of the curve, in the points P, Q respectively. The line PQ envelops a conic having a double contact with the given curve at the points A, B.

Indeed, the lines AM, BM describe two projective pencils, hence their sections by the lines a, b are two projective ranges.

$$(P\cdots) \overline{\wedge} B(M\cdots) \wedge A(M\cdots) \overline{\wedge} (Q\cdots).$$

Consequently the line PQ envelops a conic tangent to a and b. To the point (ab) considered as an element of a and b in turn correspond, in the ranges $(Q\cdots)$ and $(P\cdots)$, the points B and A, these points are therefore the points of contact of a and b with the envelope, which proves the proposition.

If for A, B are taken some remarkable points of the conic, special cases of this general proposition are obtained. For example, if A be the point at infinity of a parabola, and B its vertex, the

proposition takes the following form: *The diameter passing through a variable point of a parabola, meets the tangent at the vertex in the point P. The parallel through P to the line joining M to the vertex of the parabola, envelops another parabola having the same vertex and the same axis as the given curve.*

The proposed problem is another special case of this general proposition, namely when both A and B are at infinity.

The duals of the three propositions are, in order:

The points of intersection of two fixed tangents to a given conic, with a variable tangent to the same curve, are projected from the points where the fixed tangents touch the conic. The point of intersection of the two projecting lines describes a conic having a double contact with the given curve.

From a variable point of the tangent at the vertex of a given parabola, are drawn the diameter and the tangent to the curve. The point of intersection of the diameter with the parallel to the tangent through the vertex of the curve, describes a parabola having the same axis and the same vertex as the given curve.

The parallels to the asymptotes of a given hyperbola drawn through the points of intersection of the latter lines with a variable tangent to the curve, intersect in a point whose locus is an hyperbola having the same asymptotes as the given curve.

Also solved by O. S. ADAMS, CLARA L. BACON, J. W. CLAWSON, A. M. HARDING, HORACE OLSON, PAUL CAPRON, G. W. HARTWELL, R. M. MATHEWS, and N. P. PANDYA.

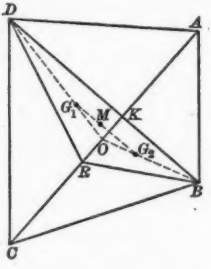
490. Proposed by ELMER E. MOOTS, University of Arizona.

In any quadrilateral $ABCD$, let AC and BD be the diagonals intersecting at K. On AC, lay off CR equal to AK. Join B and R. Connect the middle point G of BR with D. On GD lay off GM equal to $\frac{1}{3}GD$. Show that M is the center of gravity of the quadrilateral.

SOLUTION BY A. M. HARDING, University of Arkansas.

It is evident that M is the center of gravity of the triangle BDR. Hence it will be sufficient to prove that the triangle BDR and the quadrilateral $ABCD$ have the same center of gravity.

Let O be the mid-point of RK, then it will also be the mid-point of CA. Then G_1 is the center of gravity of the triangles RDK and CDA, and G_2 is the center of gravity of the triangles RBK and CBA where $OG_1 = \frac{1}{3}OD$ and $OG_2 = \frac{1}{3}OB$.



Since

$$\frac{\triangle RDK}{\triangle RBK} = \frac{\triangle CDA}{\triangle CBA},$$

is follows that the center of gravity of the triangle BDR will also be the center of gravity of the quadrilateral $ABCD$.

Also solved by J. W. CLAWSON, O. S. ADAMS, and N. P. PANDYA.

491. Proposed by N. P. PANDYA, Sojitra, India.

In a triangle $mx = b$ and $nx = c$, determine a relation between m, n, x, A and s, and solve it for x.

SOLUTION BY J. A. COLSON, Searsport, Maine.

Since $b = mx$ and $c = nx$, we have $\sin^2 A/2 = (s-b)(s-c)/bc = (s-mx)(s-nx)/mnx^2$. Hence, $mnx^2 \sin^2 A/2 = s^2 - (m+n)sx + mnx^2$, or $mnx^2 \cos^2 A/2 - (m+n)sx + s^2 = 0$.

Solving this quadratic for x, we have

$$x = \frac{(m+n)s \pm s\sqrt{\{(m+n)^2 - 4mn\cos^2 A/2\}}}{2mn\cos^2 A/2}.$$

492. Proposed by FRANK V. MORLEY, Student, Haverford College.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle, and let the symmedian point of the triangle formed by omitting a_i be s_i. Prove that the four points s_i have the same diagonal triangle as the four points a_i.

SOLUTION BY J. E. ROWE, Pennsylvania State College.

We choose that system of homogeneous coördinates in which the coördinates of a point are proportional to $\alpha/a : \beta/b : \gamma/c$, where a, b, c are the lengths of the sides of the reference triangle and α, β, γ the lengths of the $\perp$'s from the sides to the point. It may easily be shown that in this system of coördinates the equation of the circle circumscribing the reference triangle is

$$x_2x_3 + x_1x_3 + x_1x_2 = 0. \tag{1}$$

Let the coördinates of the four a's be

$$a_1 \equiv b_1, b_2, b_3; \quad a_2 \equiv 1, 0, 0; \quad a_3 \equiv 0, 1, 0; \quad a_4 \equiv 0, 0, 1.$$

There is evidently no loss of generality in this selection of the a's, and they will all lie on (1) if only

$$b_2b_3 + b_1b_3 + b_1b_2 = 0. \tag{2}$$

The coördinates of P_1 the intersection of the lines a_1a_3 and a_2a_4 are $b_1, 0, b_3$; similarly the coördinates of P_2 the intersection of the lines a_1a_4 and a_2a_3 are $b_1, b_2, 0$; and the coördinates of P_3 the intersection of the lines a_1a_2 and a_3a_4 are $0, b_2, b_3$. That is, $P_1P_2P_3$ is the diagonal triangle of the a's.

The equations of the tangents to (1) at the points a_i are

$$\begin{aligned}
T_1 &\equiv (b_2+b_3)x_1 + (b_1+b_3)x_2 + (b_1+b_2)x_3 = 0,\\
T_2 &\equiv \qquad\qquad\quad x_2 \quad + \quad x_3 = 0,\\
T_3 &\equiv \quad x_1 \qquad\qquad + \quad x_3 = 0,\\
T_4 &\equiv \quad x_1 \quad + \quad x_2 \qquad\quad = 0.
\end{aligned}$$

The tangents to (1) at two of the points a intersect in a point, and this point and a third a determine a line. Any set of three a's yields three such lines which are concurrent through the symmedian point of the three a's. In this way we find that the coördinates of the symmedian points are

$$s_1 \equiv 1, 1, 1; \quad s_2 \equiv b_1, b_1 + 2b_2, b_1 + 2b_3; \quad s_3 \equiv b_2 + 2b_1, b_2, b_2 + 2b_3; \quad s_4 \equiv b_3 + 2b_1, b_3 + 2b_2, b_3.$$

By reason of (2) the determinant

$$\begin{vmatrix} b_1 & b_1 + 2b_2 & b_1 + 2b_3 \\ b_3 + 2b_1 & b_3 + 2b_2 & b_3 \\ b_1 & 0 & b_3 \end{vmatrix} = 0.$$

Hence, the points s_2, s_4, and P_1 are collinear. In the same way it may be shown that s_1, s_3, and P_1 are collinear. From the symmetry of the coördinates of the P's and the s_i it follows that $s_2s_3P_2$, $s_1s_4P_2$, $s_1s_2P_3$, and $s_3s_4P_3$ are collinear sets of three points, and this shows that $P_1P_2P_3$ is the diagonal triangle of the s_i.

Also solved by J. W. CLAWSON and J. W. HASLEY.

CALCULUS.

410. Proposed by J. A. BULLARD, U. S. Naval Academy.

(*a*) Find the area of the loop of the curve $x^{2q+1} + y^{2q+1} = (2q+1)ax^qy^q$. (For $q = 1$ we have the folium of Prob. 379.)

(*b*) Find the area between the curve and its asymptote. (From Johnson's *Integral Calculus.*)

SOLUTION BY THE PROPOSER.

The required integration is simplified by the use of polar coördinates. If we let $y = mx$ and express the equation in parametric form we have merely to note that the parameter m is $\tan\theta$ and to substitute in the integral $\frac{1}{2}\int r^2\,d\theta$. Thus in the case of the folium (see p. 343, Vol. XXII) we have

$$A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{1}{2}\int_0^{\infty} x^2 dm = \frac{9a^2}{2}\int_0^{\infty} \frac{m^3 dm}{(1+m^3)^2} = -\frac{3a^2}{2}\left[\frac{1}{(1+m^3)}\right]_0^{\infty} = 3a^2/2.$$

(*a*) The above equation becomes in polar coördinates

$$r = \frac{(2q+1)a\tan^q\theta\sec\theta}{1+\tan^{2q+1}\theta};$$

or in parametric form

$$x = \frac{(2q+1)am^q}{1+m^{2q+1}}, \qquad y = \frac{(2q+1)am^{q+1}}{1+m^{2q+1}},$$

where $m = \tan\theta$. The loop is generated when θ varies from 0 to $\pi/2$. Thus

$$A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{(2q+1)^2a^2}{2}\int_0^{\pi/2} \frac{\tan^{2q}\theta\sec^2\theta\,d\theta}{(1+\tan^{2q+1}\theta)^2}$$

$$= \frac{(2q+1)^2a^2}{2}\int_0^{\infty} \frac{m^{2q}dm}{(1+m^{2q+1})^2} = -\frac{(2q+1)a^2}{2}\left[\frac{1}{1+m^{2q+1}}\right]_0^{\infty} = \frac{(2q+1)a^2}{2}.$$

(*b*) The equation of the asymptote is $x + y = (-1)^q a$ or

$$r = \frac{(-1)^q a\sec\theta}{1+\tan\theta}.$$

The asymptote forms a triangle of area $a^2/2$ with the coördinate axes, and this triangle separates the remainder of the required area into two equal parts. That part in the second quadrant when q is odd but in the fourth quadrant when q is even, is given by

$$A_1 = \frac{1}{2}\int_{3\pi/4}^{\pi} (r_1^2 - r_2^2)d\theta = \frac{a^2}{2}\int_{-1}^{0}\left[\frac{1}{(1+m)^2} - \frac{(2q+1)^2m^{2q}}{(1+m^{2q+1})^2}\right]dm = \frac{a^2}{2}\left[-\frac{1}{1+m} + \frac{2q+1}{1+m^{2q+1}}\right]_{-1}^{0}$$

$$= \frac{a^2}{2}\left[\frac{2q+m-m^2+m^3-\cdots+m^{2q-1}-m^{2q}}{1+m^{2q+1}}\right]_{-1}^{0}$$

$$= \frac{a^2}{2}\left[\frac{2q-(2q-1)m+(2q-2)m^2-\cdots+2m^{2q-2}-m^{2q-1}}{1-m+m^2-\cdots-m^{2q-1}+m^{2q}}\right]_{-1}^{0}$$

$$= \frac{a^2}{2}\left[2q - \frac{(2q+1)(2q)/2}{2q+1}\right] = \frac{qa^2}{2}.$$

Then

$$A = 2A_1 + a^2/2 = \frac{(2q+1)a^2}{2}.$$

Thus the area of the loop is equal to the area between the curve and its asymptote.

In the case of the folium ($q = 1$) the coördinate axes trisect the area between the curve and its asymptote.

Also solved by O. S. ADAMS, HORACE OLSON, PAUL CAPRON, and A. M. HARDING.

MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

II. Solution by William Hoover, Columbus, Ohio.

Let the radii of gyration of the shell and sphere, k', k'', be given by

$$k'^2 = \frac{2}{5}\frac{R^5 - r^5}{R^3 - r^3},$$

$$k''^2 = \frac{2}{5}r_1^2;$$

C_0T_0 the radius of the shell to the tangent point on the inclined plane initially; E_0 the common point initially of the inner surface of the shell and sphere; CT_0 the position of C_0T_0 after any time from the beginning of motion of the system; D the center of the sphere at the same time t; CT the radius of the shell to the tangent point T of shell and inclined plane at the same instant; φ = angle TCT_0; DE = the radius of the sphere to the tangent point of the inner surface of the shell and the sphere; θ = the angle DE makes with the vertical through D;

$$s = TT_0 = CC_0 = R\phi;$$

CH = a perpendicular from C upon C_0E_0 cutting the latter at H;

$$r - r_1 = r;$$

C_0 the coördinate origin; C_0C the x-axis; C_0T_0 the y-axis; x, y the coördinates of D; then $\angle C_0CH = \alpha$; $\angle C_0CD = \pi/2 - \theta + \alpha$; and then, F being the foot of the perpendicular from D upon CC_0,

$$x = s - CF = R\varphi - r'\cos DCF = R\varphi - r'\sin(\theta - \alpha);$$

$$y = DF = r'\sin DCF = r'\cos(\theta - \alpha).$$

The dynamic conditions for the motion of the sphere can be most clearly indicated by noticing that the initial point E_0 remains in contact with the inner surface of the shell, while the sphere has an angular velocity $\dot\theta$, $\dot\varphi$ being that of the shell.

Let M, m, be the masses of the shell and of the sphere; T, V, the kinetic energy, and potential energy; then the kinetic potential equation for the system is

$$T = \tfrac{1}{2}M(k'^2\dot\varphi^2 + \dot s^2) + \tfrac{1}{2}m(\dot x^2 + \dot y^2 + k''^2\dot\theta^2)$$

$$= Mgs\sin\alpha + mg(r'\cos\theta + s\sin\alpha) + C = V.$$

But

$$\dot x = R\dot\varphi - r'\cos(\theta - \alpha)\dot\theta,$$

$$\dot y = -r'\sin(\theta - \alpha)\dot\theta,$$

and

$$\dot s = R\dot\varphi,$$

whence, on substitution and reduction,

$$T = \tfrac{1}{2}\{M(k'^2 + R^2) + mR^2\}\dot\varphi^2 + \tfrac{1}{2}m\{-2Rr'\cos(\theta - \alpha)\dot\theta\dot\varphi + (r'^2 + k''^2)\dot\theta^2\}$$

$$= g\{(M + m)R\varphi\sin\alpha + mr'\cos\theta\} + C = V.$$

The Lagrangian equations

$$\frac{d}{dt}\frac{dT}{d\dot\varphi} - \frac{dT}{d\varphi} = \frac{dV}{d\varphi},$$

$$\frac{d}{dt}\frac{dT}{d\dot\theta} - \frac{dT}{d\theta} = \frac{dV}{d\theta},$$

applied to the last result, give after simplifications,

$$\{(M+m)R^2+Mk'^2\}\ddot{\varphi}-mRr'\cos(\theta-\alpha)\ddot{\theta}+mRr'\sin(\theta-\alpha)\dot{\theta}^2=g(M+m)R\sin\alpha,$$

$$Rr'\cos(\theta-\alpha)\ddot{\varphi}-(r'^2+k'^2)\ddot{\theta}=gr'\sin\theta.$$

Eliminating $\ddot{\varphi}$,

$$[(r'^2+k'^2)\{(M+m)R^2+Mk'^2\}+mR^2r'^2\cos^2(\theta-\alpha)]\ddot{\theta}-mR^2r'^2\sin(\theta-\alpha)\cos(\theta-\alpha)\dot{\theta}^2$$
$$=-gr'[\{(M+m)R^2+Mk'^2\}\sin\theta+(M+m)R^2\sin\alpha\cos(\theta-\alpha)].$$

Multiplying by $2\dot{\theta}$ and integrating

$$[(r'^2+k'^2)\{(M+m)R^2+Mk'^2\}+mr'^2R^2\cos^2(\theta-\alpha)]\dot{\theta}^2$$
$$=gr'[2\{(M+m)R^2+Mk'^2\}\cos\theta-2(M+m)R^2\sin\alpha\sin(\theta-\alpha)]+C',$$

which is of the same general form as (7), p. 351, this MONTHLY for November, 1916.

NUMBER THEORY.

235. Proposed by W. D. CAIRNS, Oberlin College.

Prove that $n=1$ is the only positive integer for which n^4+4 is a prime.

SOLUTION BY WM. E. PATTEN, Government Institute of Technology, Shanghai, China.

$$n^4+4=(n^4+4n^2+4)-4n^2=(n^2+2)^2-(2n)^2=(n^2+2n+2)(n^2-2n+2).$$

Therefore, n^4+4 is a prime, if at all, only for those values of n which make either $n^2+2n+2=1$, or $n^2-2n+2=1$, since each of the factors of n^4+4 given above is integral in value when n is integral, and both are positive when n is positive.

(1) When $n^2+2n+2=1$, then $n=-1$.

(2) When $n^2-2n+2=1$, then $n=+1$. When $n=+1$, then $n^4+4=5$, a prime.

Therefore, n^4+4 is a prime for $n=1$, and for no other positive integral values of n.

Also solved by ELMER SCHUYLER, FRANK IRWIN, HORACE OLSON, ELIJAH SWIFT, H. H. CLARK, ELIZABETH B. DAVIS, NORMAN ANNING, L. G. WELD, and the PROPOSER.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy+z=\square,\quad yz+x=\square,\quad \text{and}\quad xz+y=\square.$$

SOLUTION BY ARTEMAS MARTIN, LL.D., Washington, D. C.

Assume $x=n^2$, $y=(n+1)^2$; then the given equation becomes

$$n^2(n+1)^2+z=\square,\quad (n+1)^2z+n^2=\square,\quad n^2z+(n+1)^2=\square.$$

Put

$$n^2(n+1)^2+z=a^2.$$

Assume $a=n^2+n+b$, and the last equation becomes

$$n^2(n+1)^2+z=a^2=(n^2+n+b)^2,$$

from which we immediately find

$$z=b(2n^2+2n+b).$$

Substituting in

$$(n+1)^2z+n^2=\square,$$

we have

$$b(n+1)^2(2n^2+2n+b)+n^2=\square=c^2,$$

which is satisfied by $b = 2$; for then

$$c^2 = 4n^4 + 12n^3 + 17n^2 + 12n + 4 = (2n^2 + 3n + 2)^2,$$

and $z = 4(n^2 + n + 1)$. This value of z, with the assumed values, $x = n^2$, $y = (n+1)^2$, satisfies all the proposed conditions.

$$xy + z = n^2(n+1)^2 + 4(n^2 + n + 1) = (n^2 + n + 2)^2,$$

$$yz + x = 4(n+1)^2(n^2 + n + 1) + n^2 = (2n^2 + 3n + 2)^2,$$

$$xz + y = 4n^2(n^2 + n + 1) + (n+1)^2 = (2n^2 + n + 1)^2.$$

If $n = 1$, then $x = 1$, $y = 4$, $z = 12$.
If $n = 2$, then $x = 4$, $y = 9$, $z = 28$.
If $n = 3$, then $x = 9$, $y = 16$, $z = 52$.

And so on, indefinitely.

The values of x, y, z just found will also satisfy the conditions

$$xy + x + y = \square, \quad xz + x + z = \square, \quad \text{and} \quad yz + y + z = \square.$$

Also solved by ELIZABETH B. DAVIS and H. N. CARLETON.

237. Proposed by NORMAN ANNING, Chilliwack, B. C.

Prove that for three numbers x, y, z,

$$9\Sigma(x-y)^4 = \Sigma(2x - y - z) = 2\square.$$

SOLUTION BY E. F. CANADAY, University of South Dakota.

This problem is evidently misprinted. If we write it

$$9\Sigma(x-y)^4 = \Sigma(2x - y - z)^4 = 2\square,$$

a solution is possible. To prove

$$9[(x-y)^4 + (y-z)^4 + (z-x)^4] = (2x - y - z)^4 + (2y - z - x)^4 + (2z - x - y)^4 = 2\square,$$

we put

$$(x-y) = a, \quad (y-z) = b, \quad \text{and} \quad (z-x) = -(a+b).$$

Then

$$9[a^4 + b^4 + (-a-b)^4] = (2a+b)^4 + (b-a)^4 + (-a-2b)^4 = 9(2a^4 + 4a^3b + 6a^2b^2 - 4ab^3 + 2b^4) = 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4 + b^4 - 4ab^3 + 6a^2b^2 - 4a^3b + a^4 + a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 = 2[9(a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4)] = 18a^4 + 36a^3b + 54a^2b^2 + 36ab^3 + 18b^4 = 2\square.$$

$$2[3(a^2 + ab + b^2)]^2 = 2\square.$$

Also solved by the PROPOSER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kansas.

REPLIES.

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

Readers interested in the above question will be glad to learn that a better and more complete article than that contemplated as an answer to the question

is soon to appear in the *Annals of Mathematics*. The paper is being prepared by Professor L. E. Dickson, of the University of Chicago, and will be a somewhat extended account of the more important results in proper historical setting.

Reference may also be made to A. Fleck's six-page article on Fermat's last theorem in Auerbach and Rothe's *Taschenbuch für Mathematiker und Physiker*, 3. Jahrgang, 1913 (Teubner), pp. 103–109; Benno Lind's forty-three-page article, "Über das letzte Fermatsche Theorem" in *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft $XXVI_2$, pp. 21–65, 1910; and Lipke's four-page review of Lind's article (*Bull. Amer. Math. Soc.*, Vol. XVIII, pp. 194–198) in which references are given to the criticisms of Lind's work published in *Archiv der Mathematik und Physik*.

33. Under what conditions or to what extent is Mr. Iwerson's construction a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the MONTHLY, pp. 354, 355. The following corrections should be made: In the figure, B' should be B. In the last two lines, Ox should be OY, and Oy should be OX.

REPLY BY PAUL CAPRON, U. S. Naval Academy, Annapolis, Md.

In the figure above referred to, let $XO = OX' = a$, $YO = OY' = b$. Then $X'A = a$, $X'B = (a^2 - b^2)/\sqrt{a^2 + b^2}$. Let $k = AB$ and $l = NN' = NR$. Then $k = a - (a^2 - b^2)/\sqrt{a^2 + b^2}$, and $l = 2a - k = a + (a^2 - b^2)/\sqrt{a^2 + b^2}$. Since $X'N'P'$ and $N'NR$ are equilateral triangles,

$$RO = (l - k)\frac{\sqrt{3}}{2} = \frac{a^2 - b^2}{\sqrt{a^2 + b^2}} \cdot \sqrt{3}, \qquad \text{and} \qquad l - RO = a - \frac{a^2 - b^2}{\sqrt{a^2 + b^2}}(\sqrt{3} - 1).$$

In order that the arc drawn with R as center and l as radius (through N and N') may pass through Y, $l - RO$ must be equal to b. This is the case (if $b < a$) when and only when $a = b\sqrt{3}$.

Let $b/a = x$ (if the eccentricity is e, $e^2 + x^2 = 1$). The proportional error in the length of the minor axis is

$$E_1 = \frac{1}{b}[l - RO - b] = \frac{1 - x}{x}\left[1 - \frac{1 + x}{\sqrt{1 + x^2}}(\sqrt{3} - 1)\right].$$

The proportional errors in the radii of curvature at the ends of the axes are: at the end of the major axis,

$$E_2 = \frac{a}{b^2}\left(k - \frac{b^2}{a}\right) = \frac{a^2 - b^2}{b^2}\left(1 - \frac{a}{\sqrt{a^2 + b^2}}\right) = \frac{1 - x^2}{x^2}\left(1 - \frac{1}{\sqrt{1 + x^2}}\right);$$

at the end of the minor axis,

$$E_3 = \frac{b}{a^2}\left(l - \frac{a^2}{b}\right) = \frac{a-b}{a^2}\left[\frac{b(a+b)}{\sqrt{a^2+b^2}} - a\right] = (1-x)\left[\frac{x(1+x)}{\sqrt{1+x^2}} - 1\right].$$

$$\frac{dE_1}{dx} = \left[(\sqrt{3}-1)\frac{1+3x^2}{(1+x^2)^{3/2}} - 1\right], \qquad \frac{dE_2}{dx} = \frac{1}{x^3}\left[\frac{2+3x^2-x^4}{(1+x^2)^{3/2}} - 2\right],$$

$$\frac{dE_3}{dx} = \frac{1-3x^2-2x^4}{(1+x^2)^{3/2}} + 1.$$

Of the six variables, E_2 and dE_2/dx vanish for no real value of x between 0 and 1; the values, aside from 0 and 1, which cause the variables to vanish, are given by the following equations:

$$E_1 = 0; \qquad (x - \tfrac{1}{3}\sqrt{3})(x - \sqrt{3}) = 0,$$

$$\frac{dE_1}{dx} = 0; \qquad x^6 + 3(6\sqrt{3} - 11)x^4 + 3(4\sqrt{3} - 7)x^2 + (2\sqrt{3} - 3) = 0,$$

$$\frac{dE_2}{dx} = 0; \qquad x^4 - 10x^2 - 7 = 0,$$

$$E_3 = 0; \qquad x^4 + 2x^3 - 1 = 0,$$

$$\frac{dE_3}{dx} = 0; \qquad 4x^6 + 11x^4 + 2x^2 - 9 = 0.$$

It is said that architects find it troublesome to draw a shapely oval opening; the ellipse is the most satisfactory curve in itself, but there is much labor involved in making offsets for several parallel curves in the design of the mouldings. With Mr. Iwerson's approximation, it would merely be necessary, with four fixed centers, to use appropriately lengthened radii. The most serious defect in the shapeliness of the broken line is the abrupt discontinuity in the curvature at the four points where the radius of curvature is altered in the ratio k/l.

$$\frac{k}{l} = \frac{[\sqrt{1+x^2} - (1-x^2)]^2}{x^2(3-x^2)}.$$

(If $x < \frac{1}{2}$, $\frac{k}{l} = x^2\left(\frac{3}{4} + \frac{x^2}{12} + \frac{107}{400}x^4 - \frac{101}{200}x^6\right)$, nearly enough to give three decimal places.)

When $x = \frac{1}{\sqrt{3}}$, $(E_1 = 0)$, $\frac{k}{l} = 0.268$; when $x = \frac{3}{4}$, $\frac{k}{l} = \frac{13}{27}$; when $x = \frac{1}{2}$, $\frac{k}{l} = 0.197$.

For consecutive values of x, at intervals of 0.1:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
k/l	0.000	0.008	0.030	0.068	0.123	0.197	0.291	0.412	0.559	0.752	1.000

In order to estimate this error on somewhat the same basis as the errors E_1, E_2, E_3, the second of the following tables includes, under the caption $\pm E_4$, the value of

$$\frac{l - \frac{1}{2}(l+k)}{\frac{1}{2}(l+k)} = -\frac{k - \frac{1}{2}(l+k)}{\frac{1}{2}(l+k)} = \frac{1 - k/l}{1 + k/l}.$$

The first of the following tables shows certain critical values of E_1, E_2, E_3 and their derivatives; the second shows consecutive values of E_1, E_2, E_3, $\pm E_4$ and the eccentricity.

e	x	E_1	$\frac{dE_1}{dx}$	E_2	$\frac{dE_2}{dx}$	E_3	$\frac{dE_3}{dx}$
1.000	0.000	$+\infty$	$-\infty$	0.500	0.000	−1.000	2.000
0.817	0.577	0.000	−0.147	0.268	−0.642	−0.089	0.380
0.696	0.717					0.000	0.099
0.693	0.721	−0.008	0.000				
0.515	0.857					0.030	0.000
0.000	1.000	0.000	+0.035	0.000	−0.586	0.000	−0.414

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
E_1	$+\infty$	1.789	0.555	0.207	0.073	0.018	−0.004	−0.008	−0.007	−0.004	0.000
E_2	0.500	0.491	0.466	0.426	0.375	0.317	0.254	0.188	0.123	0.060	0.000
E_3	−1.000	−0.801	−0.518	−0.439	−0.243	−0.165	−0.071	−0.008	+0.025	0.027	0.000
$\pm E_4$	1.000	0.985	0.942	0.872	0.781	0.671	0.549	0.418	0.283	0.141	0.000
e	1.000	0.995	0.980	0.954	0.917	0.866	0.800	0.714	0.600	0.484	0.000

The constructions made from these data show that Mr. Iwerson's approximation is very close for a medium eccentricity, e. g. for $e = 0.92$, $x = 0.4$.

It is possible to calculate in advance a length to be used instead of b for OY, so that the circle with center R shall pass through the end of the minor axis.

If $b/a = x$, $\frac{1}{2}(1-x)(\sqrt{3}-1) = p$, $OY = ya$, where

$$y^4 - (2+p^2)y^2 + (1-p^2) = 0;$$

or $Oy = a \tan \phi$, where $\cos 2\phi \cdot \sec \phi = p$.

This might be worth while in case e is large, E_1 objectionable and E_2 and E of no consequence.

As to what errors are tolerable and what objectionable, that is of course entirely a matter of circumstances.

DISCUSSIONS.

I. RELATING TO THE ORDER OF OPERATIONS IN ALGEBRA.

BY N. J. LENNES, University of Montana.

§ 1. *The Rules as Given in the Books.*

Subtraction and division are defined as the inverse operations of addition and multiplication. The commutative and associative laws of addition and multiplication are, therefore, extended in the same manner to both subtraction and division. In the case of addition alone or of multiplication alone it is agreed that, when no symbols of aggregation occur, the operations are to be performed from left to right. Thus, $a + b + c$ means $(a + b) + c$, and $a \times b \times c = (a \times b) \times c$. Without such an understanding the associative laws would have no meaning.

The Commutative and Associative Laws. In case symbols of addition and subtraction both occur (and no other symbols), it is agreed that each symbol applies only to the term immediately following it, and that the operations are to be performed from left to right.

Thus, $8 - 2 + 4 = (8 - 2) + 4 = 10$, and *not* $8 - (2 + 4) = 2$. From this usage it follows that terms connected by $+$ and $-$ signs may be *commuted*, but they may not be *associated*, except when a $+$ sign precedes the group in question. Thus, $8 + 4 - 2 = 8 + (4 - 2)$, but $8 - 4 - 2$ is not equal to $8 - (4 - 2)$.

In case the signs of multiplication and division occur with no signs of addition and subtraction intervening, and in case no symbols of aggregation are used, then it is likewise agreed (in the theoretical development in the books) that each symbol applies only to the factor (or divisor) immediately following it, and that the operations are to be performed in order from left to right.

Thus, $8 \div 2 \times 4 = (8 \div 2) \times 4 = 16$, and *not* $8 \div (2 \times 4) = 1$. As in the case of addition and subtraction, it results from this agreement that the commutative law applies to the operations of multiplication and division, while the associative law does not apply, except when the sign $\times$ precedes the group in question.

That is, $8 \div 4 \times 2 = 8 \times 2 \div 4$, but $8 \div 4 \times 2$ is not equal to $8 \div (4 \times 2)$. As remarked by Chrystal, under these conventions, the associative and commutative laws for addition and subtraction are *formally identical* with these laws for multiplication and division. (*Text Book of Algebra*, Part I, page 17.)

Following this theoretical development most of the current text-books give a rule like the following:

A series of operations involving multiplication and division alone shall be performed in the order in which they occur from left to right.

§ 2. *Actual Usage.*

The Above Rule Contrary to Actual Usage. The rule stated above is agreed to by practically all those writers on algebra who make any mention of the matter at all. Chrystal gives a detailed development and writers on elementary algebra have in general followed him. It would, however, follow from this rule for carrying out multiplications and divisions in order from left to right, that

$$9a^2 \div 3a = (9a^2 \div 3) \times a = 3a^3.$$

But I have not been able to find a single instance where this is so interpreted. The fact is that the rule requiring the operations of multiplication and division to be carried out from left to right *in all cases,* is not followed by *anyone.* For example, in case an indicated product follows the sign ÷ the whole product is *always* used as divisor, except in the theoretical statement of the case.

Writers meet the situation in different ways:

(*a*) Some always use the fractional form to indicate division, this being equivalent to a symbol of aggregation. Thus, $ab/cd = (ab) \div (cd)$.

(*b*) Some write out the words in full, thus: "divide this expression by that expression."

(*c*) Some use the sign ÷ to mean that the whole product, following the sign ÷, shall be the divisor.

The most important exception to (*c*) occurs in the development of the theory in such a text as Chrystal. In Chrystal, $\div u \times v$ is sometimes used to mean $(\div u)v$. In such cases, however, the notation $\div u \times v$ and not $\div uv$ is used.

Chrystal in one case writes $2^2 \div 3^2 \times 5^2 = (2^2 \div 3^2) \times 5^2$. (Note the sign × to indicate multiplication.) He also writes $pa/pb = pa \div (pb)$. (Note the parenthesis.) This comes nearer consistency than is usually the case. However, in no case does Chrystal write $9a^2 \div 3a$ as the equivalent of $(9a^2 \div 3) \times a$. He overcomes the difficulty by *never using the sign ÷ with a product after it.*

The followers of Chrystal have too often blindly copied his theory, but have not taken pains, as he did, to avoid inconsistency in usage. Examples of such inconsistency in theory and usage could be multiplied *ad infinitum.* One text, which has been in very wide use, states (in developing the theory)

$$60 - 40 \div 5 \times 3 - 20 = 60 - \frac{40}{5} \times 3 - 20,$$

but on the next page we read:

$$10bc \div 12a = \frac{10bc}{12a}.$$

The Established Usage. When an indicated product follows the sign ÷ the whole product is, by overwhelming preponderance of actual usage, to be regarded as the divisor. Hence, the true rule as to the order of operations when both

multiplications and divisions are involved is not the one stated above, but the following:

All multiplications are to be performed first and the divisions next.

That is, $9a^2 \div 3a = 3a$ and not $3a^3$.

The multiplications may be taken in any order, but the divisions are to be taken in the order in which they occur from left to right.

That is, the associative law holds for the former but not for the latter.

Thus, $3 \times 5 \times 2 = (3 \times 5) \times 2$ or $= 3 \times (5 \times 2)$; but, $16 \div 4 \div 2 = (16 \div 4) \div 2$ and does not $= 16 \div (4 \div 2)$.

Compare the corresponding rules for addition and subtraction in § 1.

Mathematical Idioms. It might be agreed that, for the sake of simplicity and logical coherence, the past tense of the verb *to drink* should be *drinked*, but even so, English speaking people would continue to say *drank*, and not drinked. Precisely, for the same reason, all who know anything about the language of algebra regard $9a^2 \div 3a$ as equal to $3a$ and not $3a^3$, and, therefore, the rule just given is the correct one as determined by actual usage. When a mode of expression has become wide-spread, one may not change it at will. It is the business of the lexicographer and grammarian to record, not what he may think an expression should mean (no matter how far-fetched the usual or idiomatic usage may seem), but what it is *actually understood to mean by those who use it.* The language of algebra contains certain idioms and in formulating the grammar of this language we must note them. For example, that $9a^2 \div 3a$ is understood to mean $3a$ and not $3a^3$ is such an idiom. The matter is not logical but historical.

II. RELATING TO AN EXTENSION OF WILSON'S THEOREM.

By Elizabeth Brown Davis, U. S. Naval Observatory.

From Wilson's theorem we have the congruence,

$$(p-1)! + 1 \equiv 0, \pmod{p},$$

which may be written,

$$(p-1)(p-2)! + 1 \equiv 0, \pmod{p}. \tag{1}$$

Subtracting (1) from $p(p-2)! \equiv 0, \pmod{p}$, we have

$$(p-2)! - 1 \equiv 0, \pmod{p}.$$

This may be written

$$(p-2)(p-3)! - 1 \equiv 0, \pmod{p}. \tag{2}$$

Subtracting (2) from $p(p-3)! \equiv 0, \pmod{p}$, we have

$$2(p-3)! + 1 \equiv 0, \pmod{p},$$

or

$$2(p-3)(p-4)!+1\equiv 0, \quad (\text{mod. } p). \tag{3}$$

Subtracting (3) from $2p\,(p-4)!\equiv 0, \quad (\text{mod. } p)$, we have

$$3!(p-4)!-1\equiv 0, \quad (\text{mod. } p).$$

Proceeding in like manner, we obtain successively,

$$4!(p-5)!+1\equiv 0, \quad (\text{mod. } p),$$

$$5!(p-6)!-1\equiv 0, \quad (\text{mod. } p),$$

. . . . . .

. . . . . .

. . . . . .

$$\left(\frac{p-1}{2}\right)!\left(\frac{p-1}{2}\right)!\pm 1\equiv 0 \quad (\text{mod. } p),$$

the constant term being $+1$ when $(p-1)/2$ is even and -1 when $(p-1)/2$ is odd. Hence the theorem:

If p is prime, and a is any integer less than $p-1$, then

$$a!(p-1-a)!+(-1)^a\equiv 0, \quad (\text{mod. } p).$$

Wilson's theorem is the special case $a=0$, of which the above is the more general form, it being understood that $0!=1!/1=1$, and that

$$(-1)^0=(-1)^1/(-1)=+1.$$

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

At the College of the City of New York, Dr. P. H. LINEHAN has been promoted to an assistant professorship of mathematics.

Dr. CORA B. HENNEL has been promoted from an instructorship to an assistant professorship of mathematics at Indiana University.

Dr. DANIEL BUCHANAN has been made professor of mathematics and astronomy at Queen's University, Kingston, Ontario.

Professor H. L. RIETZ, of the University of Illinois, has been appointed a member of the joint committee of the American Association of University Professors with the trustees of the Carnegie Foundation for the Advancement of Teaching to report upon the proposed changes in the scope of the foundation.

"The relation of mathematics to the natural sciences" is the subject of a six-page discussion in *Science*, December 15, 1916, by Professor T. E. MASON, of Purdue University.

Professor C. SMITH, head master of Sidney Sussex College, Cambridge, died on November 13, 1916, at the age of seventy-two years. Professor SMITH was known to a large number of teachers of mathematics in America through his very excellent series of college texts on algebra, conic sections, and solid analytic geometry.

In *School and Society*, December 16, 1916, Professor G. A. MILLER, of the University of Illinois, has an interesting paper on "History and use of mathematical text-books." After an introductory discussion of the general features of good and bad text-books, Dr. MILLER devotes some space to the use made of text-books, incidentally touching different methods of teaching. He also emphasizes the importance of accuracy and clearness, and the value of historical notes in mathematical text-books.

The thirty-eighth meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on December 22 and 23, 1916, at which eleven research papers were presented by representatives of the following universities: Chicago, Illinois, Minnesota, Nebraska, Purdue and Wisconsin, and Rose Polytechnic Institute. The members present enjoyed an informal dinner together at the Quadrangle Club on Friday evening. Among topics informally discussed were (1) The contract between the Mathematical Association of America and the *Annals of Mathematics* for the enlargement of that journal and the publication in it of expository and historical articles; (2) the need in this country of careful consideration of the whole question of the history of mathematics and of combined and systematic effort in developing investigation in this line; and (3) the desirability of holding ourselves in readiness to assist the publishers of the *Revue Semestrielle* and the *Fortschritte* in case it becomes necessary, on account of the war conditions, in order to continue the publications.

The twenty-third annual meeting of the American Mathematical Society was held at Columbia University, New York, on December 27–28, 1916. Thirty-three papers were presented by representatives from twelve educational institutions including two from Canada. The session of Thursday afternoon was a joint meeting of the American Mathematical Society, the Mathematical Association of America, Section A (Mathematics and Astronomy) of the American Association, and the American Astronomical Society. At this meeting Professor E. W. BROWN, of Yale University, President of the American Mathematical Society, delivered his retiring address on "The relation of mathematics to the natural sciences," and the vice-presidential address, Section A of the American Association, on "Derivation of orbits—theory and practice," which was to have been delivered by Professor A. L. LEUSCHNER, of the University of California, was read by Professor M. W. HASKELL, of the University of California.

At the holiday meeting of the American Association for the Advancement of Science held in New York City, great interest was manifested in the programs of Section D, Engineering. On Wednesday forenoon a joint session of Section D and Section I was devoted to discussion of the social and economic question of "The advisability of adopting the metric standard of weights and measures in the United States." The program of Section D for Thursday morning was given to the presentation of papers on "Sanitary engineering"; in the afternoon of the same day a program of nine papers was provided on the general subject of "Highway engineering education"; and on the evening of the same day a joint program was devoted to the subject of "Highway engineering." On Friday the general program of Section D was continued, and in the evening the program of the section was concluded by a joint session of Section D with the American Society of Civil Engineers, the American Society of Mechanical Engineers, the American Institute of Mining Engineers, and the American Institute of Electrical Engineers, at which the topic for discussion was "The inter-relationship of engineering and pure science." In the program of Friday afternoon, Professor D. J. McADAM, of Washington and Jefferson College, presented a paper on "Mathematical education for civil engineers."

In the report of the meeting of the Kansas Section in the January issue, the Secretary of the Section inadvertently gave the wrong name as new Chairman. It should have been Professor B. L. Remick of the Kansas Agricultural College. (State.)

The editor of "Notes and News" for the MONTHLY would greatly appreciate the coöperation of members of the Mathematical Association in supplying him with items suitable for insertion in these columns. Kindly send us notices of promotions, resignations, deaths, departmental activities, mathematical clubs, local programs, summer session schedules, and notes on important mathematical contributions of general interest. Please send all such notices to D. A. ROTHROCK, Bloomington, Indiana.

NOTES ON THE ANNUAL MEETING OF THE ASSOCIATION.

The truly national character of the Mathematical Association of America is strongly shown in the List of Officers and Charter Members just published and distributed to all subscribers to the MONTHLY. The geographical distribution of members shows clearly the wisdom of holding the first two national meetings of the Association in the East, and the attendance upon these meetings, in Cambridge and in New York, has fully justified the selection of these meeting places. A second look at the directory will show equally clearly the desirability of holding the next national meetings in the Middle West. The Council in New York made provision toward this end by appointing a committee to act in connection with the American Mathematical Society with respect to holding a joint summer meeting in 1917 somewhere west of New York, and by voting to hold the next

annual meeting in December, 1917 in Chicago. These meetings should draw large numbers from all parts of the country and especially from the Middle West and South. The decision has since been made to hold the summer meeting in Cleveland.

The Association was exceedingly fortunate in the accommodations provided for its meetings at the Massachusetts Institute of Technology and at Columbia University. Hamilton Hall, which is the headquarters of the department of mathematics at Columbia University, proved admirably adapted to Association purposes, both for the larger gatherings and for committees and council meetings. As a side attraction, nothing could have been more interesting than the collection of portraits and medals of mathematicians belonging to Professor David Eugene Smith and put on exhibition by him between the sessions on Friday. Special credit is due to the committee on program and arrangements for the orderly and smooth progress of all matters pertaining to the meetings.

One matter in connection with meetings of the Association, whether national or sectional, deserves careful consideration, namely, the opening of all sessions sharply at the time announced. At Cambridge this was done rather successfully; in New York the result was not so successful, owing partly to uncontrollable circumstances and partly to the distractions arising from the many interesting meetings in progress on the campus. It might be a worthy New Year's resolution for the Association to establish a definite policy of starting all its meetings, including the meetings of sections, exactly on time, and of holding definitely to the program schedule in all respects.

The interest in establishing sections still grows apace. The Minnesota Section, reported in this issue, was duly organized and held its first meeting early in December. The Maryland Section has just been organized and was admitted by the council at the New York meeting. Hearty coöperation and much enthusiasm were reported in connection with this section. The Committee on Sections has also authorized the establishment of a section in Kentucky. Preliminary steps were taken at a small gathering in Chicago on December 23 toward the organization of a section in Illinois and more definite plans are soon to be put into action. Other sections are under consideration in various parts of the country. The Council Committee on Sections is likely to become one of the most important in the near future, not only in respect to the organization of new sections but also in coördinating the work of the sections, in stimulating their activities, and in preparing and distributing types of programs and discussions, and possibly in providing representative speakers to visit the section meetings, to give assistance in local plans, and to stir up activity and enthusiasm.

The plan of coöperation with the Annals of Mathematics, which had already been unanimously approved by a mail vote of the Council, was still further perfected in joint conferences between the committee of the Association and the editors of the Annals. It was agreed that the plan should go into operation with the final number of the present volume of the Annals, which is to be issued early in May and is to contain an expository paper by Professor L. E. Dickson,

of the University of Chicago, on Fermat's last theorem and other matters of historical interest in connection with algebraic number theory. A communication setting forth the details of this plan will be issued early in March by the editors of the Annals and the committee of the Association. Only warm commendation of this plan was heard on every hand at the New York meeting. An informal referendum showed that between sixty and seventy persons were ready to subscribe to the Annals under the half-rate offer to be extended to members of the Association. This communication will be sent to all members of the Society as well as to the members of the Association.

About fifty scientific societies are affiliated with the American Association for the Advancement of Science. Once in four years at the annual meeting of the American Association there is held a convocation of these affiliated societies, such as took place in New York City during the last week of December, 1916. As an inducement to members of the affiliated societies to join the American Association, the entrance fee of five dollars was remitted by a special order during the year 1916. This privilege has now been further extended by a special resolution at the New York meeting, to include all new members of any affiliated society who may be elected to membership in the American Association within one year of their admission to the affiliated society. This privilege would then be applicable to the charter members of the Mathematical Association of America up to April first, 1917, since the charter membership list closed on April first, 1916. The dues in the American Association are three dollars per year and all members receive the weekly journal *Science*.

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THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME XXIV MARCH, 1917 NUMBER 3

DIRECTED ANGLES IN ELEMENTARY GEOMETRY.

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In many of the theorems of the modern elementary geometry, the treatment of angles seems somewhat unsatisfactory. In the statement of theorems, one is often confronted with the dilemma of a choice between an inaccurate statement, and one so verbose and involved as to be unwieldy. Again, many proofs, as given in the texts,[1] are insufficient because they apply only to particular positions of the figure. A very common example is the following. If A, B, C, D are four points on a circle, the angles ABC, ADC are equal or supplementary, according as B and D are on the same side of AC, or on opposite sides. This theorem is repeatedly used in proofs; but in a given case, when we know only that the points are concyclic, and have no data as to their order on the circle, how are we to decide which of the two possibilities is the correct one? Apparently the usual custom is to draw a single figure, and decide by inspection of the figure, trusting that the proof so obtained can be modified to fit all possible figures. Not only is such a method entirely unscientific, but in cases where the figure is at all complicated, the determination of the number of possibilities and the corresponding modifications of the proof are practically impossible.

As a simple illustration, let us consider Simson's theorem, so-called. "If from any point P on the circumcircle of the triangle ABC, PX, PY, PZ be drawn perpendicular to the sides, the points X, Y, Z will be collinear." (This statement, and the following proof, are taken from Lachlan, *l. c.*, § 120.)

"Join ZX, YX. Then since the points P, X, Z, B are concyclic, the angle PXZ is the supplement of the angle ABP. And since P, Y, C, X are concyclic, the angle YXP is the supplement of the angle YCP, and is equal to the angle ABP, because P, C, A, B are concyclic."

[1] Such books as Lachlan, *Modern Pure Geometry;* Casey, *A Sequel to Euclid;* McClelland, *Geometry of the Circle;* etc., are here referred to.